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Pricing with Splines

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Abstract

We apply the exponential affine pricing principle to the family of skewed Laplace historical distributions. The risk-neutral distribution is shown to belong to the same family and a closed form pricing formula for european option is derived. This formula is a direct competitor of the Black-Scholes formula and involves location and tail parameters. This approach is extended to exponential affine splines and to a multiperiod framework.

Keywords : Stochastic Discount Factor, Laplace Distributions, Derivative Pricing, Splines, Markov Chain.

Valorisation avec des fonctions splines
Résumé

On applique le principe du facteur d'escompte exponentiel affine à la famille des lois de Laplace asymétriques. On montre que la probabilité risque-neutre appartient à la même famille et on en déduit une formule explicite pour la valorisation d'options européennes. Cette formule est une alternative à la formule de Black-Scholes et met en jeu un paramètre de position et un paramètre de queue. Cette approche est généralisée aux distributions de type splines exponentiels affines et au cas dynamique.

Mots clés : Facteurs d'escompte stochastique, Lois de Laplace asymétriques, Valorisation d'actifs contingents, Splines, Chaînes de Markov.

JEL number : C1, C5, G1

1. Introduction

The standard for option pricing is the Black-Scholes approach [Black, Scholes (1973)], which assumes i.i.d. gaussian geometric stock returns, continuous trading and derives an analytical formula for pricing European calls from the arbitrage free restrictions. The derivative prices and the associated risk neutral probability basically depend on the underlying historical volatility (and not on the historical mean). However it is known that the Black-Scholes specification is misspecified both for return dynamics and for pricing derivatives with different characteristics. Typically the implied Black-Scholes volatility surfaces are not flat and vary with the day and the environment.

Different solutions have been proposed in the literature to reduce the misspecification errors, that is to get call prices with independent variations with respect to the strike, the maturity and the date. The extensions of the basic model can be classified according to the assumptions introduced on the two components of a pricing models, which are the historical distribution and the stochastic discount factor (s.d.f.).

i) Parametric historical distribution-parametric sdf

A first direction consists in extending the parametric dynamic model for the underlying asset price and in deriving the new corresponding parametric valuation formulas. For instance the Black-Scholes model has been extended by introducing stochastic volatility [3], or jumps [4]. These models are generally written in continuous time and provide coherent specifications for analyzing return dynamics and cross-sectional derivative pricing. However the introduction of a non traded random factor creates an incomplete market framework. The incompleteness, that is the multiplicity of admissible s.d.f., is solved by imposing a parametric specification of the risk premium corresponding to this unobservable factor. It can be assumed constant (and unknown), but very often is taken equal to zero as in the well-known Hull-White formula.

ii) Nonparametric sdf

[3]see e.g. Hull, White (1987), Hull (1989), Chesney, Scott (1989), Melino, Turnbull (1990), Stein, Stein (1991), Heston (1993), Ball, Roma (1994).

[4]see e.g. Merton (1976), Ball, Torous (1985), Bates (1996).

Alternatively the practitioners often perform a direct nonparametric analysis of the state prices based on derivative prices. For each date they study how the call prices depend on the strike and the maturity. They can consider directly the price surface or equivalent characteristics. Standard ones are 1) the state price density which provides the Arrow-Debreu prices and is deduced from the second order derivative of the call price with respect to the strike [see Breeden, Litzenberger (1978), Banz, Miller (1978)]; 2) the surface of Black-Scholes implied volatilities obtained by inverting the Black-Scholes formula with respect to the volatility. By neglecting the historical distribution, they implicitely assume a weak relationship between the historical and risk neutral distributions, that is a nonparametric sdf. In practice the state price or implied volatility surfaces are smoothed by nonparametric approaches. Ait-Sahalia, Lo (1998) apply kernel smoothing to observed call prices and deduce the state price density as a by-product[5]. Other authors propose direct approximations of the state price density. For instance the risk neutral distribution can be approximated by mixture of distribution [6] [7]. or by means of Hermite expansions. In the latter approach, it is possible to estimate day by day parameters measuring the weights of polynomials of degree one, two, three, four... in this expansion. They are generally interpreted as implied mean, volatility, skewness and kurtosis.

The limitation of these nonparametric approaches is due to the number of liquid derivative assets. To get accurate estimators they require a large number of highly traded derivatives, with an appropriate distribution of the associated strikes. For a given day and a given underlying asset, these numbers are generally between 5 and 20, with a clustering of traded strikes close to moneyness. Thus the cross-sectional asymptotic theory generally developped with these approaches cannot apply.

Finally note that some authors [8] can be interested in testing structural equilibrium models. For this purpose they focus on the sdf which is generally estimated by moment method from data on returns, consumption,... Since

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[5]However their approach assumes that the call prices depend in a deterministic way of the asset price.

[6]see e.g. Bahra (1996), Campa, Chang, Reider (1997), Melick, Thomas (1997).

[7]see e.g. Jarrow, Ruud (1982), Madan, Milne (1994), Abken, Madan, Ramamurtie (1996)

[8]see e.g. Bansal, Hsich, Viswanathan (1993), Bansal, Viswanathan (1993), Cochrane (1996), Chapman (1997), Dittmar (2002).

they are concerned by neither historical pdf, or the state price density, the results are difficult to use for derivative pricing.

iii) Nonparametric historical distribution parametric sdf.

A nonparametric specification of the state prices can also be derived with a nonparametric historical distribution and a parametric sdf. The advantage of such a specification is to correspond to available data. The underlying asset is generally liquid, and the associated return data can be used to estimate nonparametrically the historical distribution. Once this distribution is known, the small number of parameters defining the risk correction are calibrated on observed derivative prices. Such approaches have been developed rather early as direct extensions of the standard Black-Scholes formula.

For instance it is possible to consider a continuous time model and to assume that the infinitesimal drift and volatility functions are unknown deterministic functions drift and [9], of time. The model still assumes a complete market framework, Girsanov theorem provides the unique admissible sdf and the risk neutral distribution depends on the volatility only (called local volatility). The local volatility can be estimated directly from the return data on the underlying asset. It can also be estimated from derivative data, by using the interpretation of local volatility from partial derivatives of the call price with respect to strike and time to maturity [see Dupire (1994)]. This second estimation technique is not very accurate due to the small number of liquid derivatives.

Instead of deterministic drift and volatility functions, it is also possible to assume drift and volatility depending on the return and to still derive the unique sdf by Girsanov theorem. The unknown functional parameters, that are the drift and volatility, can be estimated in various ways. For instance Ait-Sahalia (1996) assumes a linear drift and deduce a nonparametric estimator of the volatility from a kernel estimator of the marginal density. The drift can also be let unconstrained and the two functional parameter estimated by nonparametric nonlinear canonical analysis based on either kernel method [Darolles, Florens, Gourieroux (2002)], or sieve method [hansen, Scheinkman, Touzi (????), Chen, Hansen, Scheinkman (????), Darolles, Gourieroux (????)].

However it is known that a (one dimensional) diffusion model implies

[9] see e.g. Merton (1973), Dupire (1994)

restrictions on return dynamics, which are not observed on available data. Just to mention a few time reversibility [10], or constraints on tail magnitude due to the normality of the brownian motion.

To avoid these constraints we focus on the historical transition pdf, instead of the local volatility functions.

The analysis will be performed in discrete time, which implies an incomplete market framework.

The return process (y_t) is a Markov process with an unknown conditional pdf $p(y_t|y_{t-1})$.

Then the dimension of incompleteness will be diminished by considering a parametric family of stochastic discount factors.

We restrict the choice by imposing an exponential-affine stochastic discount factor. This allows the use of the Esscher transformation to pass from the historical distribution to the risk-neutral one [11]. Then the s.d.f. parameters will be constrained by the arbitrage free restrictions, and the expression of the risk neutral density $q(y_t|y_{t-1})$ will be derived.

Thus both historical and risk neutral distributions are nonparametric. The aim of the paper is to leading to parallel analysis of both densities, compatible with no arbitrage restrictions. We will see that an appropriate tool is a mixture of skewed Laplace distributions, or equivalently a spline approximation of the log-densities by splines of degree one.

The plan of the paper is the following. In section 2, we review the principle of exponential-affine pricing. Then this approach is applied to a skewed Laplace conditional historical distribution of geometric return and extended to exponential-affine splines. The example of the conditional Laplace distribution is interesting as an introductory case for the exponential-splines. It is also important by itself, since the price of the European calls admit simple expressions. The pricing formula is a direct competitor of the standard Black-Scholes, and involves two types of parameters, which capture location and tail effects. The extension to the multiperiod framework is presented in Section 3. We introduce a Markov chain specification for describing the dynamics of the different spline regimes and derive the change of probability

[10] see e.g.

[11] [see e.g. Gerber, Shiu (1994), Buhlman et alii (1996), Shiryaev (1999), Darolles, Gourieroux, Jasiak (2001)].

at any maturity. Statistical inference is discussed in Section 4. Section 5 concludes.

2. The two period framework

Let us consider the two period framework and denote by r the riskfree rate between the dates t and $t+1$ and by $y = y_{t+1} = \log(S_{t+1}/S_t)$ the geometric return on the risky asset with price S_t. The aim of this section is to explain how to derive nonparametrically the state price density at horizon 1, that is how to price the European derivative written on r_{t+1}. Of course the horizon is fixed at 1 by convention, but the approach can be applied to any horizon. We first recall the principle of exponential-affine pricing initially introduced by Gerber, Shiu (1994)[see also Buhlman et alii (1996) Gourieroux-Monfort (2001), Yao (2001)]. This approach is applied to a skewed Laplace conditional historical distribution of geometric return. Then it is extended to exponential-affine splines to derive compatible spline specifications of the historical and risk-neutral densities.

2.1 Exponential-affine pricing

Let us introduce the truncated Laplace transform (or moment generating function) of the conditional distribution of the geometric return. It is defined by :

$$\psi(u,\gamma) = E[\exp(uy)\mathbb{1}_{y>\gamma}], \tag{2.1}$$

where the notation means :

$$\psi(u,\gamma) = E(\exp\{u\log(S_{t+1}/S_t)\}\mathbb{1}_{\log(S_{t+1}/S_t)>\gamma}|I_t),$$

I_t is the information available at time t for the investor and the path dependence of ψ is not mentioned for notational convenience .

The derivative asset, whose payoff $g(y)(=g(y_{t+1}))$ is written on the geometric return of the underlying asset, can be priced by means of a stochastic discount factor model [12]. The derivative price at date t is :

$$C(g) = E[Mg(y)], \tag{2.2}$$

[12][see e.g. Hansen, Richard (1987), Campbell, Lo, McKinlay, (1997) Chapter 8, Cochrane (2001), Gourieroux, Jasiak (2001), Chapter 13].

where M denotes a stochastic discount factor. In an exponential-affine framework the stochastic discount factor is restricted to [13] :

$$M = \exp[\alpha y + \beta]. \tag{2.3}$$

It is exponential-affine with respect to the geometric return $y(= y_{t+1})$[14]. Different motivations exist for the exponential affine restriction on the stochastic discount factor, which diminish the multiplicity of pricing formulas existing in this incomplete framework.

i) First the exponential-affine restrictions underlies the usual approaches based on no arbitrage restrictions, or on equilibrium theory. For instance in a continuous time framework, where the return satisfies a one-dimensional diffusion equation, the stochastic discount factor admits an exponential affine form by Girsanov theorem. Similarly let us consider a two period price exchange economy under preference restrictions [see e.g. Breeden, Litzenberger (1978), Huang, Litzenberger (1988)]. The exponential-affine form of the stochastic discount factor corresponds to power utility functions. (see Gourieroux, Monfort (2002) for a more detailed discussion).

ii) An exponential affine specification is obtained, when we look for the risk neutral distribution which is the closest to the historical one, for the entropy criterion [see Stutzer (1996) for the proof].

iii) The choice of an exponential affine sdf often leads to tractable computations, with results easy to compare with the standard Black-Scholes formula [see the examples given in Gerber, Shiu (1994)].

iv) Last, but not least we see below that it is appropriate to define spline approximations of the historical and risk neutral densities, compatible with no arbitrage restrictions.

The arbitrage-free constraints are derived by applying the pricing formula to the zero-coupon bond with payoff 1 and to the risky asset with payoff

[13] As above the time index is omitted for convenience. More explicit equations would be : $C_t(g) = E[M_{t,t+1} g(y_{t+1})|I_t]$, where : $M_{t,t+1} = \exp(\alpha_t y_{t+1} + \beta_t)$ is the stochastic discount factor for the period $t, t+1$. The coefficients α_t, β_t and the derivative price $C_t(g)$ are I_t-measurable, whereas the stochastic discount factor $M_{t,t+1}$ is I_{t+1}-measurable.

[14] The stochastic discount factor is in general not exponential-affine with respect to the current and lagged values of the return ; indeed the lagged values influence the change of probability by means of sensitivity coefficients α and β [see the previous footnote].

$\exp y = S_{t+1}/S_t$. These constraints are :

$$\begin{cases} E[M \exp r] = 1, \\ \\ E[M \exp y] = 1. \end{cases}$$

They provide the values of the risk correcting factors α, β by solving the system below [15], which depends on the untruncated Laplace transform :

$$\begin{cases} \exp(\beta + r)\psi(\alpha, -\infty) = 1, \\ \\ \exp(\beta)\psi(\alpha + 1, -\infty) = 1. \end{cases} \quad (2.4)$$

Then the price of a European call written on $\exp y$, with (moneyness) strike k and maturity one, is easily deduced. It is given by[16] :

$$\begin{aligned} C(k) &= E[M(\exp y - k)^+] \\ &= E[\exp(\alpha y + \beta)[\exp y - k]\mathbb{1}_{y > \log k}], \\ C(k) &= \exp(\beta)[\psi(\alpha + 1, \log k) - k\psi(\alpha, \log k)], \quad (2.5) \end{aligned}$$

where α, β are the solutions of system (2.4).

2.2 Pricing with Laplace distributions

Exponential-affine pricing is a general approach which can be applied to any return distribution [17]. In this section we consider the family of skewed Laplace distributions for several reasons.

i) Both historical and risk neutral distributions will belong to the Laplace family under no-arbitrage restrictions.

ii) The skewed Laplace distribution is compatible with the fat tails (exponential tails) observed on real data.

iii) Spline approximations of order 1 of the log-densities can be easily derived from this family.

[15] When the time index is taken into account, the solutions α and β are generally path dependent, like function ψ.

[16] Note that a call written on S_{t+1} with payoff $(S_{t+1} - kS_t)^+$, where k is the moneyness strike, is a multiple of the call written on the asset price $\exp y$ with payoff $(S_{t+1}/S_t - k)^+ = (\exp y - k)^+$.

[17] under tail restrictions since the truncated Laplace transform has to exist.

i) The main result

Let us consider a geometric return, whose conditional historical distribution is a skewed Laplace distribution denoted by $\mathcal{L}(b_0, b_1, c)$. The p.d.f is given by :

$$
\begin{aligned}
p(y) \quad &= \quad \frac{b_0 b_1}{b_0 + b_1} \exp[b_0(y-c)], \text{ if } y \leq c, \\
&\quad\;\; \frac{b_0 b_1}{b_0 + b_1} \exp[-b_1(y-c)], \text{ if } y \geq c,
\end{aligned}
$$

where b_0 and b_1 are strictly positive and c is a location parameter. c is the mode of the distribution, whereas b_0 and b_1 characterize the left and right exponential tails, respectively. The mean of the distribution is :
$m = c + \frac{1}{b_1} - \frac{1}{b_o}$, and the variance is : $\sigma^2 = \frac{1}{b_0^2} + \frac{1}{b_1^2}$. Note that b_0, b_1, c can be path dependent. This type of distribution fits the conditional distribution of observed returns better than the gaussian distribution. It admits fatter tails, which decrease at an exponential rate and a sharp peak at the mode, to balance the tail effect. By applying the general approach described in subsection 2.1, we get the pricing formulas below.

Proposition 1 : If the conditional historical distribution is a skewed Laplace distribution $\mathcal{L}(b_0, b_1, c)$ with $b_0 + b_1 > 1$, and if the stochastic discount factor is exponential-affine :

i) the conditional risk-neutral distribution is unique and corresponds to the skewed Laplace distribution $\mathcal{L}(b_0 + \alpha, b_1 - \alpha, c)$, with p.d.f. :

$$
\begin{aligned}
\pi(y) \quad &= \quad \frac{(b_0+\alpha)(b_1-\alpha)}{b_0+b_1} \exp[(b_0+\alpha)(y-c)], \text{ if } y \leq c, \\
&\quad\;\; \frac{(b_0+\alpha)(b_1-\alpha)}{b_0+b_1} \exp[-(b_1-\alpha)(y-c)], \text{ if } y \geq c,
\end{aligned}
$$

where α is the solution of :

$$
\exp(c-r)(b_0+\alpha)(b_1-\alpha) = (b_0+\alpha+1)(b_1-\alpha-1),
$$

such that : $-b_0 < \alpha < b_1 - 1$.

The risk neutral distribution depends on b_0, b_1 through the sum $b_0 + b_1$, only.

ii) The price of the call written on $\exp y$ with payoff $(\exp y - k)^+$ is :

$$C(k) = C_1(k) = \frac{b_0 + \alpha + 1}{(b_0 + b_1)(b_1 - \alpha)} \exp[-(b_1 - \alpha - 1)(\log k - c)], \text{ if } \log k \geq c,$$

$$C(k) = C_2(k) = 1 - k\exp(-r) + \frac{b_1 - \alpha - 1}{(b_0 + b_1)(b_0 + \alpha)} \exp[(b_0 + \alpha + 1)(\log k - c)], \text{ if } \log k \leq c.$$

iii) By the put-call parity relationship, the put prices are :
$P(k) = P_1(k) = -1 + k\exp(-r) + \dfrac{b_0 + \alpha + 1}{(b_0 + b_1)(b_1 - \alpha)} \exp[-(b_1 - \alpha - 1)(\log k - c)]$, if $\log k \geq c$,

$$P(k) = P_2(k) = \frac{b_1 - \alpha - 1}{(b_0 + b_1)(b_0 + \alpha)} \exp[(b_0 + \alpha + 1)(\log k - c)], \text{ if } \log k \leq c.$$

Proof : See Appendix 1.

The condition $-b_0 < \alpha < b_1 - 1$ ensures the existence of the stock price. It is easily checked that there is a unique solution for α, which belongs to the interval $(-b_0, b_1 - 1)$, if and only if $b_0 + b_1 > 1$, that is if the tails are in average sufficiently thin.

The Laplace pricing formulas have their own interest and can be easily compared to the standard Black-Scholes formulas, which assumes exponential-affine s.d.f, but gaussian (conditional) return distribution. First the pricing formulas are simpler and in particular they avoid the use of the cdf of the standard normal distribution. Second they depend on two parameters c and $b_o + b_1$ instead of the single volatility in the Black-Scholes case. This allows for more flexibility.

Finally it is interesting to note that the risk correction concerns the scale parameter σ in the Black-Scholes framework, and a rebalancing of the tail magnitude in the Laplace framework. This special risk correction is easily understood and is due to the tail magnitudes.

Indeed the payoff $\exp y$ of the underlying asset may be non integrable with respect to the conditional historical Laplace distribution [18]. If $b_1 < 1$, the payoff $\exp y$ is not integrable with respect to the conditional historical Laplace distribution, whereas it is integrable with respect to the conditional risk-neutral Laplace distribution, since $b_1 - \alpha > 1$. An effect of the risk correction by α is to reduce the tails in order to ensure this integrability and the existence of a finite stock price .

Remark 1 : The price of a European call written on S_{t+1} with strike K is given by : $C^* = S_t C(K/S_t)$. Generally C^*/S_t is not an homogenous function of K/S_t, since the coefficients b_0, b_1, c can be path dependent [see Garcia, Renault (1998) for a discussion of the link between homogeneity and leverage effect].

ii) Value of the call and moneyness strike

Proposition 1 provides an explicit formula for the price of the call written on $\exp y$. It is easily checked that this price is a differentiable function of k, which decreases from 1 to 0, is convex and such that the elasticity of the call price [the put price, respectively] with respect to the moneyness strike is constant for $k \geq \exp c$ [$k \leq \exp c$, respectively].

In particular the call prices satisfy simple deterministic relationships. If k, k_1, k_2 are moneyness strikes larger than $\exp c$, we get :

$$\log C(k) = \log C(k_1) + \frac{\log k - \log k_1}{\log k_2 - \log k_1}[\log C(k_2) - \log C(k_1)].$$

Remark 2 : When the parameters b_0, b_1, c are path independent, the elasticity of the call price C^* with respect to S_t is :

[18] Note that $\exp y$ is conditionally not integrable, if and only if the conditional expectation $E_t(S_{t+1})$ does not exist. In such a framework, the standard mean-variance management cannot be applied.

$$\begin{aligned}
\frac{\partial \log C^*}{\partial \log S_t} &= 1 + \frac{\partial \log C(K/S_t)}{\partial \log S_t} \\
&= 1 + \frac{\partial \log C}{\partial \log k}(K/S_t).\frac{\partial \log(K/S_t)}{\partial \log S_t} \\
&= 1 - \frac{\partial \log C}{\partial \log k}(K/S_t).
\end{aligned}$$

Therefore the condition of constant elasticity of C with respect to the moneyness strike for large k is equivalent to the condition of constant elasticity of C^* with respect to the current stock price.

Similarly the constraint (2.6) is also valid when the derivatives are written on S_{t+1}. With obvious notations, relation (2.6) becomes :

$$\log C^*(K) = \log C^*(K_1) + \frac{\log K - logK_1}{\log K_2 - \log K_1}\{\log C^*(K_2) - \log C^*(K_1)\}.$$

ii) Implied Black-Scholes Volatility

The pricing formula given in Proposition 1 can be numerically compared to the standard Black-Scholes formula. Since the call price depends on two independent parameters, that are $b_0 + b_1$ and c, instead of only one σ in the standard Black-Scholes, the Laplace pricing formula allows for implied location or tail effects. These features are easily observed on Figures 1 and 2, which provide the Black-Scholes implied volatilities for different sets of parameters b_0, b_1, c, and $r = 0$. The Laplace model is appropriate for recovering the so-called smile, smirk and sneer effects observed in practice. It is important to note that they can be recovered without introducing a time effect [as in Merton (1973), Dupire (1994)], simply by suppressing the gaussian assumptions.

[**Insert Figure 1 : Black-Scholes implied volatilities with c varying, $b_0 + b_1 = 10$ fixed**].

[**Insert Figure 2 : Black-Scholes implied volatilities with $b_0 + b_1$ varying, $c = .06$ fixed**].

iii) Value of the call and historical parameters

The patterns of the call prices as functions of c and $b_0 + b_1$ are provided in Figures 3 and 4.

[Insert Figure 3 : Call price as a function of c]

It is always difficult to understand how the call price depends on a location parameter, that is the mean in the standard Black-Scholes model and the mode c in the Laplace framework. This feature is clearly observed, when we consider the underlying stock with cash-flow $\exp y$. When the location parameter tends to $+\infty$ (resp. $-\infty$), the cash-flow tends to $+\infty$ (resp. 0), but the price remains constant equal to one. In fact when the location parameter tends to infinity the historical distribution tends to a point mass at infinity, whereas the risk neutral distribution may tend to a limit which does not correspond to this point mass. Typically for $y = -\infty, \exp y = 0$ and we expect a price for $\exp y$ equal to zero, whereas it is equal to one. Contrary to the Black-Scholes case in which the call price is independent of the mean, we observe a mean dependence in the Laplace framework. The symmetric pattern observed in Figure 3 is due to the special choice $k = 1, r = 0$, which implies $1 - k \exp -r = 0$ and identical call and put prices [19].

[Insert Figure 4 : Call price as a function of $b_0 + b_1$]

When $b_0 + b_1 = 1$, we get $b_0 + \alpha = 0, b_1 - \alpha = 1$ and the call price is equal to one. When $b_0 + b_1 \to +\infty$, there exists an underlying historical distribution such that the variance tends to zero and the stock geometric return is constant equal to the riskfree rate. Then $C(k) = \exp -r(\exp r - k)^+ = [1 - k \exp -r]^+$, where $\exp -r$ is introduced for discounting.

iv) A particular case

Finally let us consider the case $c = r$, where the mode of the (conditional) historical distribution corresponds to the riskfree return. The risk correcting factor α is the solution of :

[19] It is easily checked that the correcting factor $\alpha = \alpha[b_0, b_1, \exp(c - r)]$ satisfies : $\alpha[b_0, b_1, \exp(r - c)] = b_1 - b_0 - 1 - \alpha[b_0, b_1, \exp(c - r)]$.

$$(b_0+\alpha)(b_1-\alpha) = (b_0+\alpha+1)(b_1-\alpha-1)$$

$$\iff \quad \alpha = \frac{b_1-b_0}{2} - \frac{1}{2}.$$

By replacing in the expression of the call-prices, we get :

$$\begin{cases} C_1(k) &= \dfrac{1}{2\bar{b}}\exp[-(\bar{b}-1/2)(\log k - r)], \text{ if } \log k \geq r, \\ \\ C_2(k) &= 1 - k\exp(-r) + \dfrac{1}{2\bar{b}}\exp[(\bar{b}+1/2)(\log k - r)], \text{ if } \log k \leq r. \end{cases}$$

As mentioned above, the pricing formula depends on the single parameter $\bar{b} = \dfrac{b_0+b_1}{2}$, which measures the average tail magnitude. This parameter $\bar{b}$ has the same role than the volatility σ in the Black-Scholes model. When $\bar{b}$ increases, the average tail decreases. The derivatives of the call prices with respect to $\bar{b}$ are the analogues of the standard Black-Scholes vega. They are given by :

$$\begin{aligned} \frac{\partial C_1}{\partial \bar{b}}(k) &= -\frac{1}{2\bar{b}^2}\exp[-(\bar{b}-1/2)(\log k - r)][1+\bar{b}(\log k - r)], \text{ if } \log k \geq r, \\ \frac{\partial C_2}{\partial \bar{b}}(k) &= -\frac{1}{2\bar{b}^2}\exp[(\bar{b}+1/2)(\log k - r)][1-\bar{b}(\log k - r)], \text{ if } \log k \leq r. \end{aligned}$$

These derivatives are negative, which implies a decreasing relationship between the average tail magnitude $\bar{b}$ and the call price. By inverting the pricing formula, we can define the implied tail magnitude associated with any observed call price. The surface of implied Laplace tail magnitude contains the same information as the call-price surface.

It is interesting to consider the admissible call prices when the historical variance $\sigma^2 = \dfrac{1}{b_0^2} + \dfrac{1}{b_1^2}$ is known. Since the price is a monotonous function of $\bar{b}$, we get an interval of admissible prices, whose bounds are obtained for the values of b_0, b_1, which optimize $b_0 + b_1$ submitted to $\sigma^2 = \dfrac{1}{b_0^2} + \dfrac{1}{b_1^2}$. We easily deduce this interval, for instance when $\log k \geq r$. We get :

$$C_1(k) \in [0, \frac{\sigma}{2\sqrt{2}} \exp(-[\frac{\sqrt{2}}{\sigma} - \frac{1}{2}](\log k - r))], \text{ if } \sigma < 2\sqrt{2},$$

$$C_1(k) \in [0, 1], \text{ if } \sigma \geq 2\sqrt{2}.$$

The interval increases with σ, and is equal to $[0,1]$ in the limiting case $\sigma = 2\sqrt{2}$. The latter interval is the largest one compatible with the free arbitrage inequalities, since the constraints $0 \leq (\exp y - k)^+ \leq \exp y, \forall k$, imply $0 \leq C(k) \leq 1$.

2.3 Pricing with splines

The Laplace family distribution can be directly extended by increasing the number of exponential regimes for the density. Let us consider the p.d.f.

$$p(y) = \exp[a + yb_0 + \sum_{j=1}^{J} b_j (y - c_j)^+], \tag{2.6}$$

where a is fixed by the unit mass restriction, $c_1 < \ldots < c_J$ defines a partition of $I\!R, b_0 > 0, \sum_{j=0}^{J} b_j < 0$. This distribution is denoted by $\mathcal{S}(b_0, b_1, \ldots, b_J, c_1, \ldots, c_J)$. It is immediately noted that the specification corresponds to a spline approximation of the log-density by splines of degree 1. By increasing the number of nodes J and introducing fine partitions, we can approximate any p.d.f. [see e.g. De Boor (1978)].

It is also interesting to note that the specification corresponds to a mixture of truncated exponential distributions. More precisely, with the convention $c_0 = -\infty, c_{J+1} = +\infty$, the conditional p.d.f. can also be written as :

$$p(y) = \exp[a - A_j + B_j y], \text{ if } y \in (c_j, c_{j+1}) \text{ for } j = 0, \ldots, J, \tag{2.7}$$

where :

$$A_j = \sum_{l=1}^{j} b_l c_l \text{ (with } A_0 = 0),$$

$$
\begin{aligned}
B_j &= \sum_{l=0}^{j} b_l, \\
\exp a &= \left[\sum_{j=0}^{J} \frac{\exp(-A_j)}{B_j} (\exp B_j c_{j+1} - \exp B_j c_j) \right]^{-1}.
\end{aligned}
\tag{2.8}
$$

Thus the conditional historical distribution is a mixture of truncated exponential distributions :

$$
p_j(y) = B_j \frac{\exp B_j y}{\exp B_j c_{j+1} - \exp B_j c_j} \mathbb{1}_{(c_j, c_{j+1})}(y),
\tag{2.9}
$$

with weights :

$$
\begin{aligned}
\pi_j &= \frac{\exp(-A_j)}{B_j} (\exp B_j c_{j+1} - \exp B_j c_j) \\
&\quad [\Sigma_l \frac{\exp(-A_l)}{B_l} [\exp B_l c_{l+1} - \exp B_l c_l)]^{-1}.
\end{aligned}
\tag{2.10}
$$

Proposition 2 : If the conditional historical distribution is specified as an exponential-affine spline and if the stochastic discount factor is exponential-affine :

i) the conditional risk neutral distribution is unique and is the exponential-affine spline distribution $\mathcal{S}(b_0 + \alpha, b_1, \ldots, b_J, c_1, \ldots, c_J)$ whose p.d.f. is :

$$
\begin{aligned}
q(y) &= \exp[a^q + y(b_0 + \alpha) + \sum_{j=1}^{J} b_j (y - c_j)^+] \\
&= \sum_{j=0}^{J} \{\exp[a^q - A_j + (B_j + \alpha) y] \mathbb{1}_{(c_j, c_{j+1})}(y)\},
\end{aligned}
$$

where a^q is fixed by the unit mass restriction and α is solution of :

$$
\begin{aligned}
&\exp r \sum_{l=0}^{J} \left\{ \frac{\exp(-A_l)}{B_l + \alpha} [\exp[(B_l + \alpha) c_{l+1}] - \exp[(B_l + \alpha) c_l]] \right\} \\
= &\sum_{l=0}^{J} \left\{ \frac{\exp(-A_l)}{B_l + \alpha + 1} [\exp[(B_l + \alpha + 1) c_{l+1}] - \exp[(B_l + \alpha + 1) c_l]] \right\}.
\end{aligned}
$$

ii) The price of the call is given by :

$$\begin{aligned}
C(k) &= C_j(k) \\
&= [\sum_{l=0}^{J} \frac{\exp(-A_l)}{B_l+\alpha+1}\{\exp[(B_l+\alpha+1)c_{l+1}] - \exp[(B_l+\alpha+1)c_l]\}]^{-1} \\
& [\frac{\exp((-A_j)}{B_j+\alpha+1}\{\exp[(B_j+\alpha+1)c_{j+1}] - \exp[(B_j+\alpha+1)\log k]\} \\
&- \frac{k\exp(-A_j)}{B_j+\alpha}\{\exp[(B_j+\alpha)c_{j+1}] - \exp[(B_j+\alpha)c_j]\} \\
&+ \sum_{l=j+1}^{J} \frac{\exp(-A_l)}{B_l+\alpha+1}\{\exp[(B_l+\alpha+1)c_{l+1}] - \exp[(B_l+\alpha+1)c_l]\} \\
&- k\sum_{l=1+1}^{J} \frac{\exp(-A_l)}{B_l+\alpha}\{\exp[(B_l+\alpha)c_{l+1}] - \exp[(B_l+\alpha)c_l]\},
\end{aligned}$$

for $\exp c_j \leq k \leq \exp c_{j+1}$.

Proof : See Appendix 2.

In statistical theory the approximations by splines are usually introduced to estimate nonparametrically regression functions. The result of Proposition 2 can be used in a similar way for semiparametric pricing [20]. Indeed any conditional p.d.f. can be approximated as close as possible by an exponential affine spline, when the partition is increased. Proposition 2 says that this approximation, not only its limit when the number of observations in increased, is appropriate for derivative pricing, since it provides compatible approximations for both the historical and risk neutral distributions. These approximations can be used for cross-sectional pricing, that is for pricing at a given date, a given maturity,and any strike. The implementation is along the following lines :

[20] Other nonparametric pricing methods are discussed in Gourieroux, Monfort (2001), Darolles, Gourieroux, Jasiak (2001).

i) Fix a partition $c_1, \dots, c_J$;

ii) Estimate the parameters $b_j, j = 0, \dots, J$ from either the historical distribution, or observed derivative prices [see Section 4 for a more detailed discussion of the alternative estimation methods].

iii) Deduce the estimated risk correction α by solving the equation providing α after the replacement of $b_j, j = 0, \dots, J$ by their estimates.

iv) Reconstitute the estimated historical and risk neutral distributions by replacing $b_j, j = 0, \dots, J$ and α by their estimates.

2.4 Parametric sdf.

By selecting a stochastic discount factor which is exponential affine in the return, we considered a two parameter specification of the s.d.f. These parameters have been fixed by the no-arbitrage restrictions leading to a unique admissible s.d.f. It is possible to increase the dimension of the parameter of the s.d.f. in a very simple way, by considering for instance :

$$M(y) = \exp[\alpha_o + \alpha_1 y + \sum_{\rho=2}^{L} \alpha_2 (y - c_\rho^*)^+],$$

where c_ρ^* are given thresholds included in the partition defining the spline. The arbitrage restrictions imply two constraints on the $L+1$ parameters . Thus $L-1$ parameters are free and the set of s.d.f. is still parameterized under the arbitrage restrictions. For instance by choosing a zero threshold, we get different risk corrections for positive and negative returns, which can be used directly to create an asymmetric smile effect.

3. Dynamic exention and statistical inference

3.1 The specification

As mentioned in the introduction different specifications have been considered in the literature for derivative pricing. To be useful for practitioners, including traders, risk managers and regulators, the estimated models have to satisfy the following properties.

P. 1 Goodness of fit.

The model has to provide good fit for highly traded derivatives, especially the option which are to at the money options. The goodness of fit can diminish with the traded volume.

P. 2. Coherency.

A minimal coherency has to be introduced for the analysis of the option prices and of the price of the underlying asset. Indeed the underlying asset corresponds to a European call with zero strike. Moreover it is always the most traded derivative.

P. 3 Dynamics.

The model has to provide the dynamics of option prices. Indeed this dynamics is used for dynamic management of derivative portfolios and also to compute the Value at Risk (see e.g. Gourieroux, Jasiak (2003)].

P. 4 Absence of arbitrage opportunity.

The pricing model has to be compatible with the absence of arbitrage opportunity, that is the estimated risk neutral density has to satisfy the nonnegativity and unit mass restriction for any given maturity. Moreover the risk densities at different maturities have to be compatible, that is they have to satisfy the absence of dynamic arbitrage opportunity. The later condition is especially important to price path dependent derivatives.

P. 5 Robustness.

The results have to be robust to slight modifications of the data set, such as a small change of the set of derivatives which are considered as actively traded. Robustness is generally archived by introducing parametric constraints an sdf, which is the component past of the model which is difficult to reconstitute from data only.

P. 6 Smoothness.

The specification has to provide derivative prices or predictions which are sufficiently smooth with respect to time or environment. This can also be achieved by appropriate parametric restrictions. This is useful to avoid erratic changes in derivative portfolio allocations, or in the required capital associated with the VaR.

P. 7 Efficient use of the data.

Both types of data on underlying asset and derivative prices have to be used jointly.

The different approaches proposed in the literature can be easily compared with respect to the criteria above.

a) For instance the cross-sectional methods proposing nonparametric smoothing of the state prices have been introduced to satisfy the goodness of fit criterion and the smoothness with respect to the strike. [21] By focusing on criterion P, they lack other criterion : they are not robust and often provide very different results for successive days. This lack of robustness is essentially due to the small set of highly traded derivatives per day and to their varying structure per strike (a selectivity bias). In practice the data set are often enlerged by ad-hoc procedures which can consist in :

i) taking into account the price of not highly traded derivatives.

In such a case the selected price is often taken as a weighted average of the rather different bid and ask prices [see e.g. the discussion in Melick, Thomas (????) about the determination of the ????? price on oil market by expert's committee from the bid and ask prices].

ii) gathering the data on several successive days, chich disregard the effect of lagged prices.

iii) selecting a s.d.f. based on a market return and not on the asset return [see e.g. Ait-Sahalia (????) and the discussion in Engle, Rosenberg (1999) p. ???].

Moreover the cross-sectional approaches provide daily approximation of

[21] Examples of this approach are : Bahra (1996), Sherrick, Garcia, Timpattur (1996), Melick, Thomas (1997) based on mixtures of Log normal distributions, Jarrow, Rudd (1982), Corrado, Sn (1996) based on Edgeworth expansion around the log-normal distribution, Madan, Milne (1993), Abken, Madan, Ramamutre (1996) based on Hermite polynomial expansions, Jondeau, rockinger (2001) for a comparison of these approaches.

the state price density, without trying to related them dynamically.

Finally some standard nonparametric approaches which have been used in the derivative pricing framework are not compatible with the no arbitrage restriction. This is the case of polynomial or unconstrained spline approximations of the implied Black-Scholes volatility surface, which do not ensure the nonnegativity and unit mass condition in finite sample. the same remark holds for the nonnegativity conditions and the approximation of the state price density by Edgeworth, Gran-Charlier or Hermite expansions, in order to focus on implied skewness, implies kurtosis.

bà The other extreme approach is based on parametric specification of the underlying price dynamics and of the sdf (in the incomplete market framework). The characteristic of this approach is to focus on dynamic features (P3) and coherent framework (condtion P2). However the parametric specification can be misspecified, which implies poor goodness of fit (P) and lack of robustness (P). These drawbacks are well-know for the standard Black-Scholes, which is not able to reproduct the observed smile and ??? effects.

In section 2 we have introduced a specification in order to avoid some drawbacks of the two extreme approaches, without reducing a lot the ???? specific advantages. The dynamics which is approximated by exponential splines satisfy the following assumptions :

A.1 The information at date t includes the lagged and current values of the riskfree short term interest rate r_t, of the return y_t and of the derivative prices.

A.2 The price at date t of derivatives written on the underlying return depends on r_t, y_t only.

A.3 Under the historical probability the process (y_t) is such that the conditional distribution of y_{t+1} given $\underline{y_t}, \underline{r_t}$ depends on the past by y_t, r_t only. It is denoted by $p(y_{t+1}|y_t, r_t)$.

A.4 A stochastic discount factor at horizon 1 for derivatives written on y_t is :

$$M_{t,t+1} = \exp[\alpha_o(y_t, r_t) + \alpha_1(y_t, r_t)y_{t+1} + \sum_{l=2}^{L} \alpha_l(y_t, r_t)(y_{t+1} - c_l^*)^+],$$

$L, C_1^* = 0, C_2^*, \ldots, C_L^*$ are given.

Both the historical pdf and the sdf are nonparametric since the functions $p, \alpha_o, \alpha_1, \ldots$ are not constrained a priori. These assumptions are easily compareted with some other set of assumptions proposed in the literature. For instance in Ait-Sahalia (1996) the historical transition p corresponds to a diffusion model with affine drift, and the sdf is uniquely defined by the complete market hypothesis. In the framework above the transition can correspond to any drift and any distribution of the innovation, whereas an infinity of sdf are possible L is larger than 2.

However the nonparametric dimension is not the same for the two components in order to adjust to the lack of cross-sectional information. Indeed, for a given environment y_t, r_t, the historical pdf has still an infinite dimension, whereas the sdf becomes parametric with dimension $L+$.

3.2 Estimation method

The estimation method has to take into account both types of data on the underlying asset and on derivative prices. In practice the number of highly traded derivatives per day at horizon 1, say, is rather small, betwwen 4 and 15. Thus the number of fonctional parameters in the sdf has to be selected rather small $L = 2, 3, 4$ to avoid identification problems in some environment. However it can be noted that the number of information derivatives for a given environment y_t, r_t can be larger, since several days can correspond to this environments.

Any efficient estimation method has to mix a likelihood criterion corresponding to the observed basic returns and the historical transition, and a calibration criterion corresponding to the derivative prices and the constrained sdf. This criterion can be optimized jointly with respect to all functional parameters, or by a two step procedure by looking first to the transition, then to the sdf. The transition will be approximated by exponential splines :

$$p(y_t | \underline{y_{t-1}}, \underline{r_{t-1}}) \simeq \exp[a(y_{t-1}, r_{t-1} + y_t b_o(y_{t-1}, r_{t-1}) + \sum_{j=1}^{J} b_j(y_{t-1}, r_{t-1})(y_t - ??)$$

3.2.1 Estimation of the transition.

The functional parameters $a(.), b_o(.), b_j(.), j = 1, \ldots, J$ can be estimated by a local likelihood method. The values of the function $a(y_1, r), b_j(y, r) j = 0, \ldots J$ corresponding to given conditioning $y_{t-1}, r_{t-1} = r$ are approximated by :

$$\begin{aligned}
& (\hat{a}(y, r), \hat{b}(y, r)] \\
= \; & \arg\max_{a,b} \sum_{t=1}^{T} K(\frac{y_{t-1} - y}{h}) k(\frac{r_{t-1} - r}{h}) \log p(y_t, a, b), \\
= \; & \arg\max_{a,b} L_T(a, b; y, r),
\end{aligned}$$

where the optimization takes into account the unit mass restriction and $p(y_t; a, b) = \exp[a + y_t b_o + \sum_{j=1}^{J} b_j (y_t - c_j)^+]$.

3.2.2 Calibration step

The functional parameters of the sdf can be estimated in a second step by nonlinear least squares applied on observed derivative prices. To simplify the presentation we focus on European derivatives with residual maturity 1. At date t, L_t derivative prices are observed. The prices are denotedy by $P_{l,t} l = 1, \ldots, L_l$ and corresponds to payoffs $g_l(y_t)$, say. The pricing formula provides a (nonparametric) formula for any derivative price :

$$\begin{aligned}
P_{l,t}(a, b, \alpha) &= E_t(M_{t,t+1} g_l(y_t)] \\
&\simeq \int M_{t,t+1}[\alpha(y_t, r_t)] p(y_{t+1} | a(y_t, r_t), b(g_t, r_t)) b(g_t, r - t)) g_l(y_t) dg_t
\end{aligned}$$

Then the value $\alpha(y, r)$ associated with a given conditioning $y_t = y, r_t = r$ can be estimated by :

$$\begin{aligned}
\tilde{\alpha}(y, r) &= \arg\min_\alpha \sum_{t=1}^{T} K(\frac{y_t - y}{h}) K(\frac{r_t - r}{h}) \sum_{l=1}^{L_t} [P_{lt} - P_{lt}(\tilde{a}, \tilde{b}, \alpha)]^2 w_{lt}, \\
&= \arg\min_\alpha C_T(\alpha, \tilde{a}, \tilde{b}, y, r),
\end{aligned}$$

where w_{lt} are appropriate weights.

Several remarks can be done about this approach.

i) The use of exponential spline allows for a closed form expression of the price of the European call. Thus in practice we avoid the use of Monte-Carlo integration for computing $P_{lt}(a, b, \alpha)$.

ii) Among the derivatives are the riskfree zero-coupon bond (take ($g_l = 1$) and the underlying asset (take) ($g(y) = \exp y$).

iii) It is natural to select the weights in increasing relationship with the traded volume. In particular since the traded volumes for the risk-free bond and the underlying asset will be much larger than for the other derivatives, the calibration will imply a very small residual $P_{lt} - P_{lt}(\tilde{a}, \tilde{b}, \alpha)$ for these two assets. In the limiting case of $w_{jt} = +\infty$, we get the strict no arbitrage restrictions of Section.

3.2.3 Joint estimation

The two step estimation method is convenient to find easily estimates of both types of functional parameters. Theses estimates can be used as initial values in the optimization of the joint criterion :

$$[\hat{a}(y, r), \hat{b}(y, r), \hat{\alpha}(y, r)]$$

$$= \arg\max_{a,b,\alpha} \{L_T(a, b; y, r) - \lambda C_T(\alpha, a, b; y, r)\}$$

where λ is introduced to balance the likelihood and calibration criteria, that is the information included in the underlying return and derivative prices, respectively. If the model is well-specified, the estimators $\hat{a}, \hat{b}, \hat{\alpha}$ are also consistent. Intuitively they are also more efficient, especially for a, b, which are estimated from the whole information including derivatives. In particular we can expect a better accuracy concerning the tails of the historical transition, if European calls with large moneyness strike are observed.

5. Concluding remarks

The success of the Black-Scholes approach is due to a simple analytical formula for european call prices. However this formula is based on restrictive assumptions and may induce various mispricing. For instance the implied volatility has to be constant with the moneyness strike, whereas smile effects are often observed ; it has to be independent of the time to maturity, whereas an increasing dependence may be observed. Moreover it is varying with time and environment, since it neglects time dependency. The aim of this paper was to introduce alternative analytical formulas, which can be used to approximate the derivative prices for given date and residual maturity. We first derive a pricing formula for the skewed conditional Laplace distribution, before extending the analysis to exponential-affine splines. This leads to a semiparametric pricing approach. Finally, we introduce underlying Markov regimes in order to link the derivative prices for different dates and residual maturities.

Appendix 1 :
Pricing with Laplace distribution)

i) The truncated Laplace transform

Let us assume $\gamma > c$ and $u < b_1$; we get :

$$\begin{aligned}
\psi(u,\gamma) &= E[\exp(uy)\mathbb{1}_{y>\gamma}] \\
&= \exp(uc)E\{\exp[u(y-c)]\mathbb{1}_{y>\gamma}\} \\
&= \exp(uc)\frac{b_0 b_1}{b_0+b_1}\int_\gamma^\infty \exp[-(b_1-u)(y-c)]dy \\
&= \exp(uc)\frac{b_0 b_1}{b_0+b_1}\frac{\exp[-(b_1-u)(\gamma-c)]}{b_1-u}.
\end{aligned}$$

If $\gamma < c$, we get :

$$\begin{aligned}
\psi(u,\gamma) &= \exp(uc)\frac{b_0 b_1}{b_0+b_1}\int_c^\infty \exp[-(b_1-u)(y-c)]dy \\
&+ \exp(uc)\frac{b_0 b_1}{b_0+b_1}\int_\gamma^c \exp[(b_0+u)(y-c)]dy \\
&= \exp(uc)\frac{b_0 b_1}{b_0+b_1}\frac{1}{b_1-u} \\
&+ \exp(uc)\frac{b_0 b_1}{b_0+b_1}\frac{1}{b_0+u}\{1-\exp[(b_0+u)(\gamma-c)]\}.
\end{aligned}$$

Note that the truncated Laplace transform is defined for $u \in (-b_0, b_1)$.

ii) The arbitrage free conditions

If $-b_0 < u < b_1$ the Laplace transform is given by :

$$\begin{aligned}\psi(u,-\infty) &= exp(uc)\frac{b_0 b_1}{b_0+b_1}\frac{1}{b_1-u}+\exp(uc)\frac{b_0 b_1}{b_0+b_1}\frac{1}{b_0+u} \\ &= \exp(uc)\frac{b_0 b_1}{(b_0+u)(b_1-u)}.\end{aligned}$$

Thus the arbitrage free conditions become :

$$\begin{cases}\exp(\beta+r)\psi(\alpha,-\infty)=1, \\ \exp(\beta)\psi(\alpha+1,-\infty)=1,\end{cases}$$

$$\iff \begin{cases}\exp(\beta+r+\alpha c)\dfrac{b_0 b_1}{(b_0+\alpha)(b_1-\alpha)}=1, \\ \exp[\beta+(\alpha+1)c]\dfrac{b_0 b_1}{(b_0+\alpha+1)(b_1-\alpha-1)}=1.\end{cases}$$

In particular the risk correcting factor is the solution of the second degree equation, satisfying $-b_0<\alpha<b_1-1$:

$$\exp(c-r)(b_0+\alpha)(b_1-\alpha)=(b_0+\alpha+1)(b_1-\alpha-1).$$

It is easily checked that this equation has a unique solution in the interval $(-b_0, b_1-1)$, where the Laplace transforms $\psi(\alpha,-\infty)$ and $\psi(\alpha+1,-\infty)$ are both defined.

iii) **The price of the call**.

For $logk>c$, we get :

$$
\begin{aligned}
C(k) &= \exp\beta[\psi(\alpha+1, logk) - k\psi(\alpha, logk)] \\
&= \exp\beta\left\{\exp[(\alpha+1)c]\frac{b_0 b_1}{b_0+b_1}\frac{\exp[-(b_1-\alpha-1)(logk-c)]}{b_1-\alpha-1}\right. \\
&\quad \left. -k\exp(\alpha c)\frac{b_0 b_1}{b_0+b_1}\frac{\exp[-(b_1-\alpha)(logk-c)]}{b_1-\alpha}\right\} \\
&= \exp\beta\exp[(\alpha+1)c]\exp[-(b_1-\alpha-1)(logk-c)]\frac{b_0 b_1}{b_0+b_1}\frac{1}{(b_1-\alpha)(b_1-\alpha-1)} \\
&= \frac{b_0+\alpha+1}{(b_0+b_1)(b_1-\alpha)}\exp[-(b_1-\alpha-1)(\log k-c)],
\end{aligned}
$$

by the arbitrage free condition.

The computation is similar for $logk < c$ and provides :

$$
C(k) = 1 - k\exp(-r) + \frac{1}{b_0+b_1}\frac{b_1-\alpha-1}{b_o+\alpha}\exp[(b_0+\alpha+1)(logk-c)].
$$

iv) **Continuity of the pricing function**.

The value of the call is a continuous function of k. Indeed we get :

$$
\begin{aligned}
C_1(\exp c) &= \frac{b_0+\alpha+1}{(b_0+b_1)(b_1-\alpha)}, \\
C_2(\exp c) &= 1 - \exp(c-r) + \frac{1}{b_0+b_1}\frac{b_1-\alpha-1}{b_0+\alpha} \\
&= 1 - \frac{(b_0+\alpha+1)(b_1-\alpha-1)}{(b_0+\alpha)(b_1-\alpha)} + \frac{1}{b_0+b_1}\frac{b_1-\alpha-1}{b_0+\alpha} \\
&= \frac{b_0+\alpha+1}{(b_0+b_1)(b_1-\alpha)}
\end{aligned}
$$

The continuity property is still satisfied for the derivative of the value of the call with respect to k. Indeed the first order derivative of the pricing function is :

$$\begin{aligned}
\frac{dC_1(k)}{dk} &= -\frac{(b_1-\alpha-1)(b_0+\alpha+1)}{(b_0+b_1)(b_1-\alpha)}\frac{1}{k}\exp[-(b_1-\alpha)(\log k - c)], \\
\frac{dC_2(k)}{dk} &= -\exp(-r) + \frac{(b_1-\alpha-1)(b_0+\alpha+1)}{(b_0+b_1)(b_0+\alpha)}\frac{1}{k}\exp[(b_0+\alpha)(\log k - c)].
\end{aligned}$$

At the limiting point $k = \exp c$, we get :

$$\begin{aligned}
\frac{dC_1(\exp c)}{dk} &= -\frac{(b_1-\alpha-1)(b_0+\alpha+1)}{(b_0+b_1)(b_1-\alpha)}\exp(-c), \\
\frac{dC_2(\exp c)}{dk} &= -\exp(-r) + \frac{(b_1-\alpha-1)(b_0+\alpha+1)}{(b_0+b_1)(b_0+\alpha)}\exp(-c).
\end{aligned}$$

and :

$$\begin{aligned}
\frac{dC_1(\exp c)}{dk} &= \frac{dC_2(\exp c)}{dk} \\
\iff \quad \exp(c-r) &= \frac{(b_1-\alpha-1)(b_0+\alpha+1)}{(b_0+b_1)(b_0+\alpha)} + \frac{(b_1-\alpha-1)(b_0+\alpha+1)}{(b_0+b_1)(b_1-\alpha)} \\
&= \frac{(b_1-\alpha-1)(b_0+\alpha+1)}{(b_1-\alpha)(b_0+\alpha)},
\end{aligned}$$

which is exactly the equation defining α

v) **Risk neutral distribution**

The p.d.f. of the risk neutral distribution is still a Laplace distribution. Indeed this p.d.f. is given by :

$$\begin{aligned}
q(y) &= \exp(r)\frac{b_0 b_1}{b_0+b_1}\exp(\beta+\alpha c)\exp[(b_0+\alpha)(y-c)], \text{ if } y \le c, \\
& \quad \exp(r)\frac{b_0 b_1}{b_0+b_1}\exp(\beta+\alpha c)\exp[-(b_1-\alpha)(y-c)], \text{ if } y \ge c.
\end{aligned}$$

By using the arbitrage free condition, we get :

$$
\begin{aligned}
q(y) &= \frac{(b_0+\alpha)(b_1-\alpha)}{b_0+b_1}\exp[(b_0+\alpha)(y-c)], \text{ if } y \leq c, \\
& \frac{(b_0+\alpha)(b_1-\alpha)}{b_0+b_1}\exp[-(b_1-\alpha)(y-c)], \text{ if } y > c.
\end{aligned}
$$

Finally it is easily checked that the risk neutral distribution depends on b_0, b_1 through b_0+b_1 and c only. This property is satisfied if both $\alpha_0 = b_0+\alpha$ and $\alpha_1 = b_1 - \alpha$ depend on b_0+b_1 and c only. It is easily seen that α_0 and α_1 are solutions of the equations :

$$\exp(c-r)\alpha_0(b_0+b_1-\alpha_0) = (\alpha_0+1)(b_0+b_1-\alpha_0-1),$$

$$\exp(c-r)\alpha_1(b_0+b_1-\alpha_1) = (\alpha_1-1)(b_0+b_1-\alpha_1+1),$$

and the result follows.

Appendix 2 :
Pricing with exponential-affine splines

i) **The historical distribution**

The distribution is given by :

$$p(y) = \exp[a + b_0 y + \sum_{j=1}^{J} b_j (y - c_j)^+],$$

where the constant a is fixed by the constraint of unit mass. This p.d.f. can also be written as :

$$p(y) = \exp(a - A_j + B_j y), \text{ if } y \in (c_j, c_{j+1}),$$

where :

$$\begin{aligned} A_j &= \sum_{l=1}^{j} b_l c_l \text{ (with) } A_0 = 0), \\ B_j &= \sum_{l=1}^{j} b_l. \end{aligned}$$

Then the integral of the p.d.f. is :

$$\begin{aligned} \int_{-\infty}^{+\infty} p(y)dy &= \sum_{j=0}^{J} \int_{c_j}^{c_{j+1}} \exp(a - A_j + B_j y)dy \\ &= \exp(a) \sum_{j=0}^{J} (\exp(-A_j) \frac{\exp B_j y}{B_j}]_{c_j}^{c_{j+1}}) \\ &= \exp(a) \sum_{j=0}^{J} \frac{\exp -A_j}{B_j} [\exp B_j c_{j+1} - \exp B_j c_j]. \end{aligned}$$

We deduce the expression of the p.d.f :

$$p(y) = \sum_{j=0}^{J} \{\exp[-A_j + B_j y]\mathbf{1}_{(c_j,c_{j+1})}(y)\} \left\{\sum_{j=0}^{J} \frac{\exp -A_j}{B_j}(\exp B_j c_{j+1} - \exp B_j c_j)\right\}^{-1}$$

$$= \left\{\sum_{j=0}^{J} \frac{\exp -A_j}{B_j}(\exp B_j c_{j+1} - \exp B_j c_j)\frac{B_j \exp B_j y}{\exp B_j c_{j+1} - \exp B_j c_j}\mathbf{1}_{(c_j,c_{j+1})}(y)\right\}$$

$$\left\{\sum_{j=0}^{J} \frac{\exp -A_j}{B_j}(\exp B_j c_{j+1} - \exp B_j c_j)\right\}^{-1}.$$

ii) **The truncated Laplace transform :**

Let us assume $\gamma \in (c_j, c_{j+1})$; we get :

$$\begin{aligned}
\psi(u,\gamma) &= E[\exp(uy)\mathbb{1}_{y>\gamma}] \\
&= \int_{\gamma}^{c_{j+1}} \exp(a - A_j + B_j y + uy)dy \\
&+ \sum_{l=j+1}^{J} \int_{c_l}^{c_{l+1}} \exp(a - A_l + B_l y + uy)dy \\
&= \frac{\exp(a - A_j)}{B_j + u}\{\exp[(B_j + u)c_{j+1}] - \exp[(B_j + u)\gamma]\} \\
&+ \sum_{l=j+1}^{J} \frac{\exp(a - A_l)}{B_l + u}\{\exp[(B_l + u)c_{l+1}] - \exp[(B_l + u)c_l]\}.
\end{aligned}$$

iii) **The arbitrage free conditions**

The (untruncated) Laplace transform is given by :

$$\psi(u,-\infty) = \sum_{l=0}^{J} \frac{\exp(a - A_l)}{B_l + u}\{\exp[(B_l + u)c_{l+1}] - \exp[(B_l + u)c_l]\},$$

and the correcting factor α is solution of the equation :

$$\exp(r)\psi(\alpha,-\infty) = \psi(\alpha+1,-\infty)$$

or equivalently :

$$\exp(r)\sum_{l=0}^{J}\left\{\frac{\exp(-A_l)}{B_l+\alpha}(\exp[(B_l+\alpha)c_{l+1}]-\exp[(B_l+\alpha)c_l])\right\}$$

$$=\sum_{l=0}^{J}\left\{\frac{\exp(-A_l)}{B_l+\alpha+1}(\exp[(B_l+\alpha+1)c_{l+1}]-\exp[(B_l+\alpha+1)c_l])\right\}.$$

iv) **The risk-neutral distribution**

By multiplying the historical p.d.f by $\exp(\alpha y+\beta+r)$, we get a risk-neutral density with an exponential-affine spline representation. The limiting points of the partition $c_j, j=1,\ldots,J$ are unchanged, whereas the parameters of the truncated exponential distributions become : $B_j^q = B_j+\alpha$. Since : $B_j^q=\sum_{l=0}^{j} b_l^q$, we immediately deduce that :

$$b_0^q = b_0+\alpha, b_j^q=b_j, j=1,\ldots,J,$$

$$A_j^q = A_j, j=0,\ldots,J.$$

Thus the risk-neutral p.d.f. is :

$$\begin{aligned} q(y) &= \exp[a^q+y(b_0+\alpha)+\sum_{j=1}^{J}b_j(y-c_j)^+] \\ &= \sum_{j=0}^{J}[\exp[a^q-A_j+(B_j+\alpha)y]\mathbb{1}_{(c_j,c_{j+1})}(y)]. \end{aligned}$$

v) **The price of a call**

Let us assume $\gamma\in(c_j,c_{j+1})$; the price of a call is given by :

$$
\begin{aligned}
C(k) &= \frac{1}{\psi(\alpha+1,-\infty)}[\psi(\alpha+1,\log k) - k\psi(\alpha,\log k)] \\
&= [\sum_{l=0}^{J} \frac{\exp(-A_l)}{B_l+\alpha+1}\{\exp[(B_l+\alpha+1)c_{l+1}] - \exp[(B_l+\alpha+1)c_l]\}]^{-1} \\
& \quad [\frac{\exp(-A_j)}{B_j+\alpha+1}\{\exp[(B_j+\alpha+1)c_{j+1}] - \exp[(B_j+\alpha+1)\log k]\} \\
&- k\frac{\exp(-A_j)}{B_j+\alpha}\{\exp[(B_j+\alpha)c_{j+1}] - \exp[(B_j+\alpha)\log k]\} \\
&+ \sum_{l=j+1}^{J} \frac{\exp(-A_l)}{B_l+\alpha+1}\{\exp[(B_l+\alpha+1)c_{l+1}] - \exp[(B_l+\alpha+1)c_l]\} \\
&- k\sum_{l=j+1}^{J} \frac{\exp(-A_l)}{B_l+\alpha}\{\exp[(B_l+\alpha)c_{l+1}] - \exp[(B_l+\alpha)c_l]\}]
\end{aligned}
$$

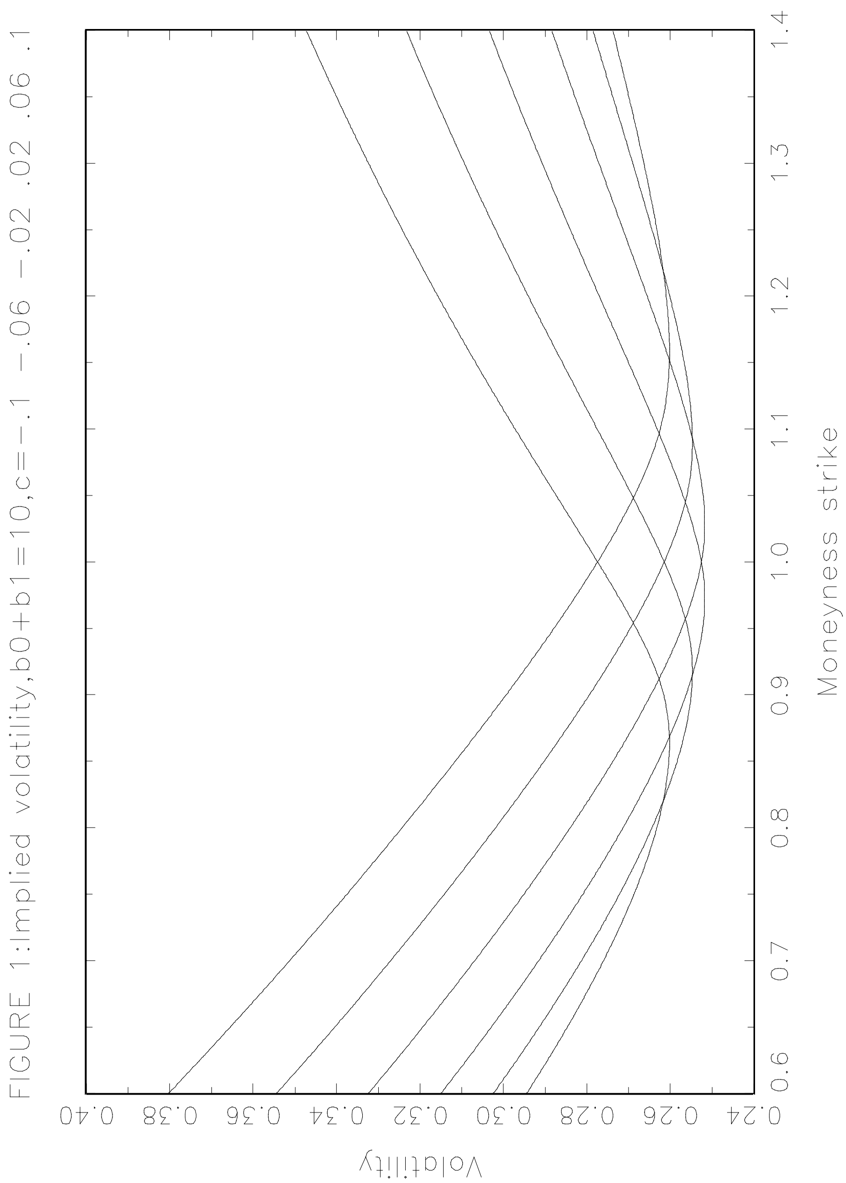


FIGURE 1:Implied volatility,b0+b1=10,c=-.1 -.06 -.02 .02 .06 .1
Volatility
Moneyness strike
0.24 0.26 0.28 0.30 0.32 0.34 0.36 0.38 0.40
0.6 0.7 0.8 0.9 1.0 1.1 1.2 1.3 1.4

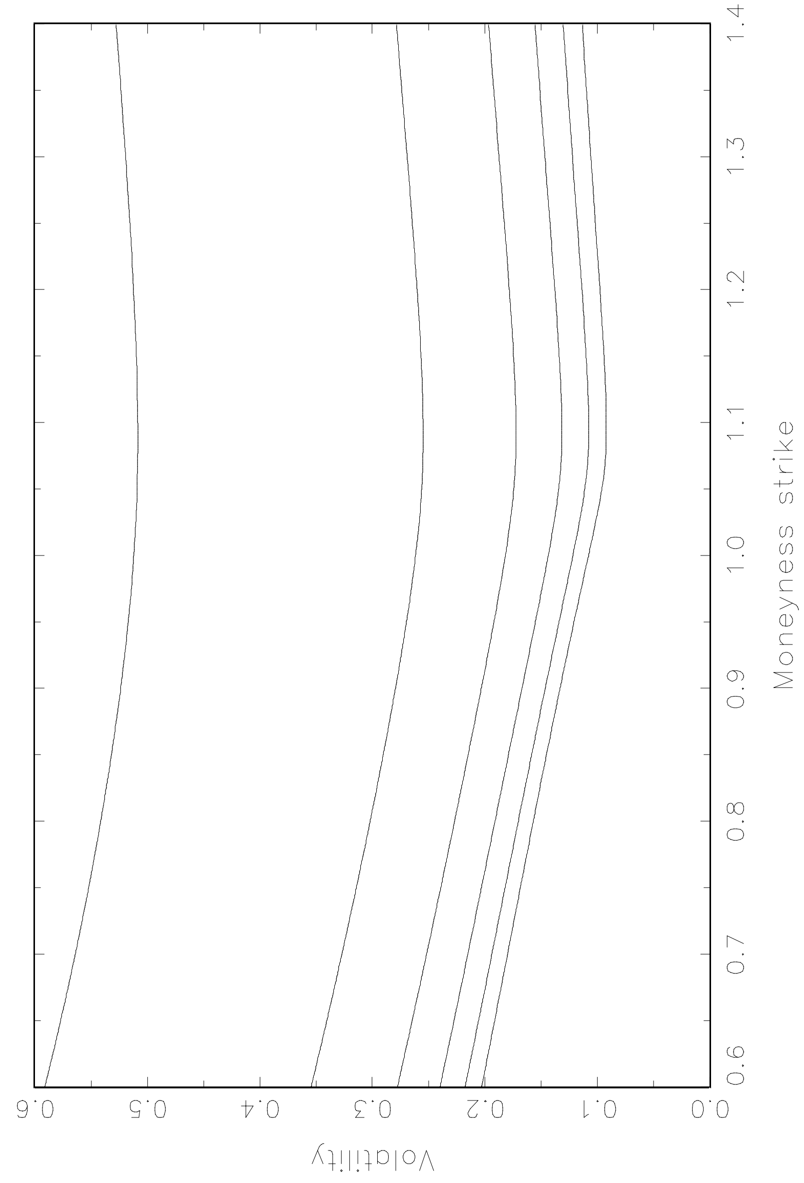
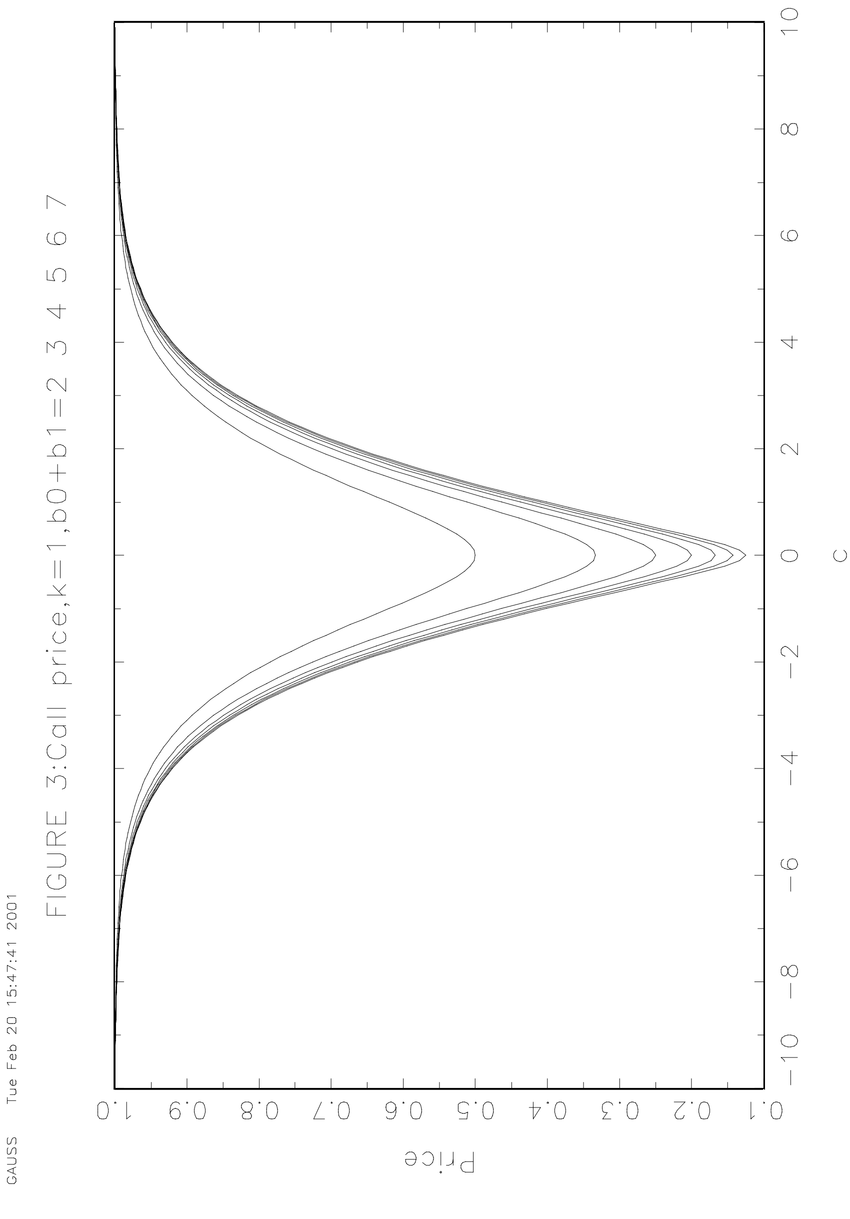
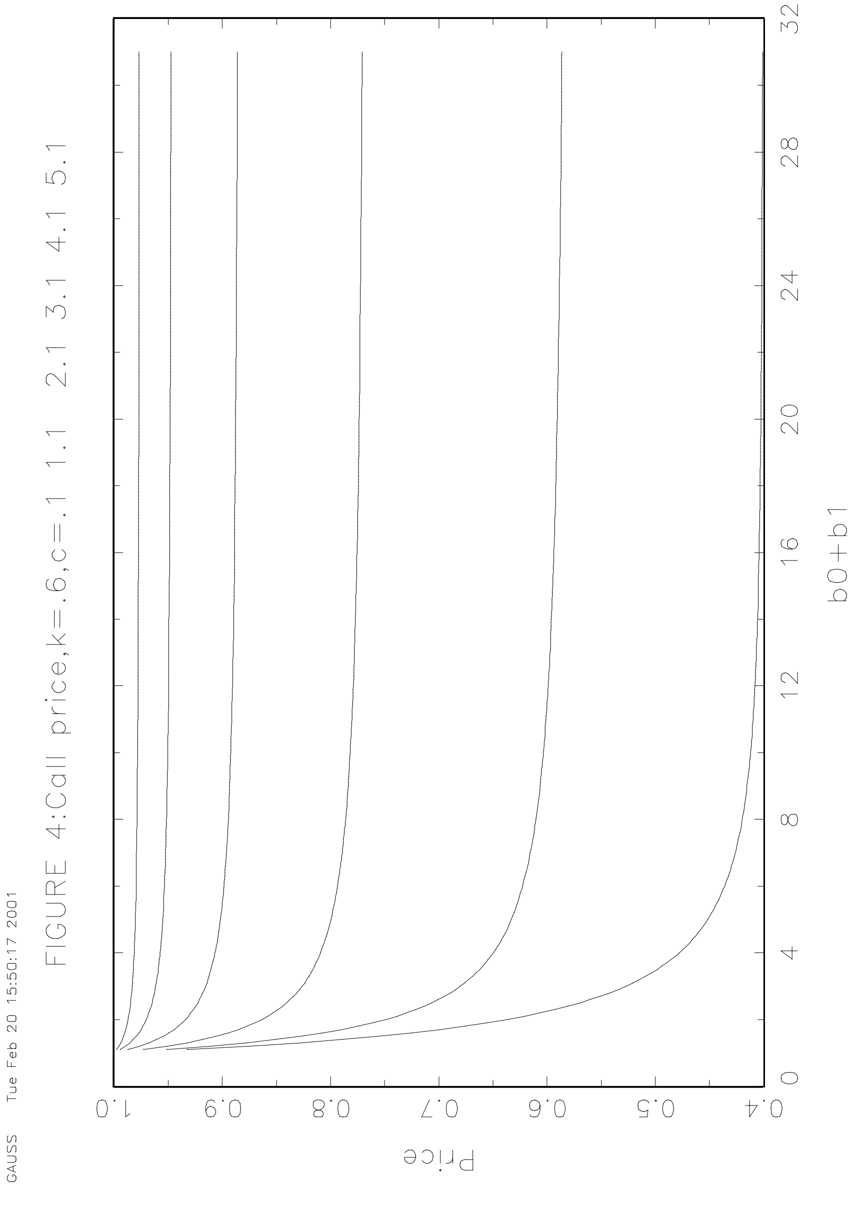


FIGURE 2:Implied volatility,c=.06 b0+b1=5 10 15 20 25 30


GAUSS Tue Feb 20 15:47:41 2001
FIGURE 3:Call price,k=1,b0+b1=2 3 4 5 6 7
Price
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
c
−10
−8
−6
−4
−2
0
2
4
6
8
10


GAUSS Tue Feb 20 15:50:17 2001
FIGURE 4:Call price,k=.6,c=.1 1.1 2.1 3.1 4.1 5.1
Price
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0
4
8
12
16
20
24
28
32
b0+b1

R E F E R E N C E S

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