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Partial Asymmetric Information and Equilibrium in a Continuous Time Model

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Abstract. This paper deals with market equilibrium under asymmetric information. We consider a model in continuous time with a single informed trader as in Back. The market is composed by three assets, one riskless and two risky. The insider has private information only on one of them. The aim of this article is to study the existence of an equilibrium when the insider is risk neutral or risk averse.
Résumé. Cet article traite des équilibres de marché avec asymétrie d'information. Comme dans l'article de Back (1992), on considère un modèle en temps continu avec un unique agent informé. Le marché est composé de trois actifs négociables, un actif sans risque et deux actifs risqués. L'agent informé ne dispose d'information que sur un des deux actifs risqués. Le but de cet article est l'étude de l'existence d'un équilibre quand l'agent informé est risque neutre ou averse au risque.

Key words: equilibrium theory ; portfolio optimization ; asymmetric information ; pricing in continuous time .

JEL Classification: G11, G12

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1 Introduction

The problem of insider trading is a well studied subject. Many papers deal with the behaviour of the prices and the portfolios when one or several agents received private information. Markets with asymmetric information has been studied with different points of view. This paper will use an extension of Kyle's model [5]. In fact, one of the main difference between works comes from the presence or not of noise traders. Bhattacharya, Reny and Spiegel [2] develop a model where this kind of agents is absent, and they are able to show that trades could collapse. In our model, if the equilibrium exists such a phenomenon cannot occur. Back [1] studied such equilibrium when the market is composed by one risk free asset and a risky one. He concludes at the existence of an equilibrium when the informed trader is risk neutral. Cho [4] shows that we can have the same kind of results when the insider is risk averse with an exponential utility. An other type of works deals with markets with more than one risky security. Caballé and Krishnan [3] shows the existence of an equilibrium in a discrete time model and Lasserre [6] proves that it is still true in a continuous time framework and when the noise traders' demand is not anymore a Brownian motion but an Ornstein Ulhenbeck process.

The aim of this paper is to study the existence of an equilibrium when there is partial asymmetric information in the market which means that the informed trader does not have necessarily private information on all assets. We suppose that agents can trade on two types of risky assets. The price of the first one is a geometric Brownian motion and all the traders know the dynamics of this price. The other is subject to asymmetric information. The market is waiting for a public release of information on this security. Only one agent, the insider, already knows this information by receiving a private signal. Excluding the informed agent, traders are of two types : market makers and noise traders. In fact, the modeling remains close to Back's framework [1].

Our purpose is to allow a feed back effect on the price of the second asset. We want to study the influence of the first security on the price formation of the second asset. In fact, we show that when the insider is risk neutral, he prefers to trade separately the two securities which is natural in a certain way since he does not want to add noise in the price formation. We may think that such a model will drive us to a possible collapse of the market, as in Bhattacharya and Reny's [2] paper since noise traders would prefer trading on the first asset. However this kind of behaviour is not permitted by our model since we suppose that the noise traders will always trade on the second asset for liquidity or hedge needs. In fact, we show that the price of the second asset depends on the price of the first one only through the coefficients of the covariance matrix of noise traders' demand. This means that the insider trades in such a way that the correlation between the two kinds of assets is exactly the same that the correlation given by the market. It appears that the informed trader does not want to reveal is information through the price of the second types of assets. However, we remark the same type of behaviour for the insider. At the equilibrium, his strategy will influence the global demand. In one hand, the global demand of the assets on which he has private information will appear as an Ornstein Ulhenbeck process regarding to the market maker. This is understandable if we consider that the informed trader does not want to reveal his signal and appears as the other traders present on the market. On the other hand, the cumulative demand is a Brownian bridge regarding to the insider. In fact, as he knows the final price (if we consider that is signal is the price of the assets when the information is released), he can invert the price to determine the associated global demand, and forced it to attain the level that realize the price he knows.

The plan of the paper is as follows. In Section 2, we expose the model. Section 3 deals with the links between our model and filtering theory. We derive the equilibrium in Section 4. The proofs of the results are in the Appendix.

2 The model

The market is composed by one risk-free asset and two risky assets. We suppose without loss of generality that the risk-free rate is zero. Moreover, the different assets can be indefinitely divisible and we assume that these assets are continuously traded. Finally, we suppose that there is no constraint on the assets as transactions costs.

We denote by $t = 1$, the announcement date of a public release information, and by $t = 0$ the present date. The information is represented by a signal $\widetilde{v}$, which can be understood as the value of one of the two risky assets after the release of information.

2.1 Type of the different agents

We consider three main types of agents.

There is a centralized market, meaning a market where a class of agents organizes the market and they can see all trades, one usually calls them market makers. In fact, we can consider that there is only one reasonable market maker who determines the prices with all the information he has. We also assume that this market maker is risk neutral, this can be justified by the fact that he has a diversified portfolio.

At the date $t = 0$, a unique agent, different from the market maker, receives a signal $\widetilde{v}$, we call him the insider. He is conscious that he is the only agent to get this extra information, and he tries to use it to maximize his utility. The market maker is also conscious that there will be a release of information at $t = 1$ and that there is only one insider. It seems natural to think that the demand of the insider depends on the signal, hence the market maker will observe the different demands on the market to determine this signal. The insider will have to hide his demand in order to not reveal his extra information. The signal $\widetilde{v}$ is not necessary something illegal, it could be the capacity of a class of agents to have precise forecasts of prices of the assets.

However, the insider's demand is not directly observed by the market maker. In fact, there exists a third class of agents. The market maker observes an aggregate demand, this third class disturbs the market maker in his reading of the insider demand. We usually call agents of this third class the noise traders. Their presence can be explained for some reasons of hedging or liquidity. As for market makers, we assume that there is a unique noise trader representing all the class.

2.2 Insider and market maker information

Let (Ω, F, P) be a probability space. For all agents, the value of risky assets is a vector V at $t=0$, except for the insider who receives the signal $\widetilde{v}$. We suppose that the signal has the following form :

$$\widetilde{v} = v + \varepsilon$$

where v is the is the real value of the random variable V after the release of information and ε is a noise. But, here, we suppose that the insider perfectly knows the value of V at time $t=1$. Hence, $\widetilde{v}$ is a realization of V. Moreover, we take an hypothesis done by Cho [4] :

Assumption 1 *We suppose that the vector $\widetilde{v}$ is a smooth function of a Gaussian random variable.*

$$\widetilde{v} = h(\Theta) \tag{1}$$

where $\left\{ \begin{array}{l} \Theta \sim \mathcal{N}(m_0, \Sigma_0) \text{ with } \Sigma_0 \geq 0 \\ h : \Re^N \to \Re^N \text{ is a smooth homeomorphism} \end{array} \right.$.

We denote respectively by X_t and Z_t the insider's demand and the noise trader's one at time t. The price vector is represented by P_t. We denote respectively

$$\begin{aligned} F^X &= \left(F_t^X, 0 \leq t \leq 1\right) \\ F^Z &= \left(F_t^Z, 0 \leq t \leq 1\right) \\ F^P &= \left(F_t^P, 0 \leq t \leq 1\right) \end{aligned}$$

the filtrations generated by the processes X, Z and P.

Definition 2 *We call cumulative demand the vector Y_t defined by :*

$$\forall t \in [0,1] \quad Y_t = X_t + Z_t \tag{2}$$

We denote by $F^Y = \left(F_t^Y, 0 \leq t \leq 1\right)$ the filtration generated by the process Y.

In fact, the market maker observes Y, but not X and Z separately, hence $F_t^M = F_t^Y$ where F_t^M is the filtration of the market maker at time t. However, $F_t^Z \nsubseteq F_t^M$ and $F_t^X \nsubseteq F_t^M$. Generally, the filtration of the market maker is larger than F_t^Y, because he has to consider some exogenous data from the market. In our case, we take them equal.

The information F_t of the insider comes from the signal received at time zero and from the observation of the randomness of the market represented by the process $(Z_t)_{t \in [0,1]}$, so

$$\forall t \in [0,1] \quad F_t = \sigma(\widetilde{v}) \vee F_t^Z \tag{3}$$

The demand process X of the insider is F_t-adapted. By the meantime, as the insider observes the prices, he knows the cumulative demand Y and so Z by (??). Finally, we obtain the two following inclusions :

$$F_t^Z \subset F_t \text{ and } F_t^Y \subset F_t$$

2.3 Rational price

As we saw before, we place ourselves in a model where there are continuous trades in continuous time. This allows the different agents to pass orders at every time and they will be executed immediately at the market price. Back [1] shows rigorously the equilibrium found by Kyle [5] but in a more general way. In fact, Back does not assume any kind of linearity of the prices with respect to the demand or the linearity of the strategies. We assume, as in Cho [4], that the prices depend on the whole path of the cumulative demand.

Definition 3 *A pricing rule is a couple (H, λ) defined by*

$$H : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$$

where H is of class $C^{1,2}\left((0,1) \times \Re^N\right)$ and

$$\forall s \in [0,1] \quad \lambda(s) = \begin{pmatrix} \lambda_1(s) & 0 \\ 0 & \lambda_2(s) \end{pmatrix}$$

with $\forall (i,s) \in \{1,2\} \times [0,1] \quad \lambda_i(s) > 0$ and $\lambda_i(.)$ is a C^1 diffeomorphism. We assume that H satisfies :

$$\begin{aligned} E\left(\int_0^1 H^2(s, Z_s)\, ds\right) &< +\infty \\ \text{and } E\left(\int_0^1 \left(\frac{\partial H}{\partial y}\right)^T \frac{\partial H}{\partial y} ds\right) &< +\infty \end{aligned}$$

We denote by $\mathcal{H}$ the set of all couples (H, λ) having the previous properties.

Definition 4 *An admissible price is a price such that $S_t = H(t, \Psi_t)$ where Ψ_t is defined by :*

$$\Psi_t = \int_0^t \lambda(s)\, dY_s \text{ and } (H, \lambda) \in \mathcal{H}. \tag{4}$$

Definition 5 *We say that the admissible price $S = (S_t, t \in [0,1])$ is rational if*

$$\forall t \in [0,1] \quad S_t = E\left[V \mid F_t^Y\right] \tag{5}$$

Remark 6 The rational price is a price such that the expectation of the benefit of the market maker conditionally to his filtration equals zero. It is the same thing to say that the prices are F_t^Y-martingales i.e. we assume that there exists a risk neutral probability associated to the market maker. We can see the prices as the results of a Bertrand's competition among the market makers.

2.4 Dynamics of the exogeneous price

We call S_t^1 the price process of the asset which we know the dynamics. We consider that its dynamics has the following form :

$$dS_t^1 = S_t^1 \left(\mu dt + \sigma dB_t^1\right) \tag{6}$$

where B_t^1 is a real Brownian motion and μ, σ are deterministic functions.

2.5 Demand of the noise trader

The modeling we take for the demand of the noise trader is essential because it determines the price's framework. In fact, if the insider wants to hide his strategy and not to reveal his information, he has to stay close to the strategy of the noise trader. We suppose that the demand process Z_t of the noise trader is an Ornstein Ulhenbeck process :

$$dZ_t = \left(a_0\left(t\right) + a_1\left(t\right) Z_t\right) dt + \varepsilon\left(t\right) dB_t \tag{7}$$

with $\forall t \in [0,1]$ $\varepsilon\left(t\right)$ is a nonsingular 2×2 matrix, and $B_t = \begin{pmatrix} B_t^1 \\ B_t^2 \end{pmatrix}$ is a bidimensional Brownian motion. We notice that the first component of the Brownian motion B_t is the real Brownian motion which appears in (**??**).

2.6 Equilibrium

We denote by W_{1^+} the final wealth of the insider after the release of information. It is a function of the insider's demand $X = (X_t)_{t\in[0,1]}$ and of the price $P = (P_t)_{t\in[0,1]}$. We assume that the insider has a utility function $U\left(.\right)$ in the Von Neumann-Morgenstein sense. The insider wants to maximize conditionally to his filtration, $(F_t)_{t\in[0,1]}$, the expectation of the utility of his final wealth $E\left[U(W_{1^+}(P,X)) \mid F_t\right]$.

Definition 7 *Let $\mathcal{X}$ be a class of strategies of the insider and $\mathcal{H}$ a class of pricing rules. Given a pricing rule (H, λ), we say that a strategy X^* is H-optimal on the class $\mathcal{X}$ if :*

$$\forall t \in [0,1] \quad \forall X \in \mathcal{X} \quad E\left[U\left(W_{1^+}\left(H,X\right)\right)/F_t\right] \leq E\left[U\left(W_{1^+}\left(H,X^*\right)\right)/F_t\right]$$

Definition 8 *We say that (H^*, λ^*, X^*) is an equilibrium on the space $(\mathcal{H},\mathcal{X})$ if it satisfies :*
(i) The market efficiency condition : $H^*\left(t, \Psi^*\right)$ is the rational price for a given strategy of the insider X^*, with $Y^* = X^* + Z$ and $\Psi_t^* = \int_0^t \lambda^*\left(s\right) dY_s^*$.
(ii) The insider's optimality condition : X^* is an optimal strategy for the given pricing rule (H^*, λ^*).

3 Representation of processes on the insider's filtration

3.1 Assumptions

As in Lasserre [6] we make two assumptions. For more details, one can refer to the previous article.

Assumption 9 *(H): We assume that*

$$Q_t(\omega) = P(\widetilde{v} \in . \mid F_t)(\omega) \ll P(\widetilde{v} \in .) = \nu(.) \quad \forall t \in [0,1] \quad p.p.\omega \in \Omega$$

Then, we consider the dynamics of Z on F_t :

$$dZ_t = \varepsilon \rho_t dt + \varepsilon d\widetilde{B}_t \tag{8}$$

where $\widetilde{B}_t$ is a F_t-Brownian motion.

Assumption 10 *(H') We assume that the signal $\widetilde{v}$ has the form*

$$\widetilde{v} = \widetilde{v}_i + cZ_1$$

where c is deterministic and $\widetilde{v}_i \perp\!\!\!\perp F^Z$. Hence, using enlarged filtration theory, the process ρ has the following form :

$$\rho_t = b_0(t, Z_1) + b_1(t, Z_1) Z_t$$

For examples, we refer to Lasserre [6].

3.2 Dynamics of the state variables

We recall that the noise trader's demand has the following form :

$$dZ_t = (a_0 + \varepsilon b_0 + (a_1 + \varepsilon b_1) Z_t) dt + \varepsilon(t) d\widetilde{B}_t \tag{9}$$

For convenience, we write : $dZ_t = (A_0 + A_1 Z_t) dt + \varepsilon(t) d\widetilde{B}_t$.

The price S_t^1 has to be written under the insider's filtration. With the assumptions we made, we directly have :

$$\begin{aligned} dS_t^1 &= S_t^1 \left((\mu + \Sigma b_0 + \Sigma b_1 Z_t) dt + \Sigma d\widetilde{B}_t \right) && (10) \\ &= S_t^1 \left((\widehat{\mu} + \Sigma b_1 Z_t) dt + \Sigma d\widetilde{B}_t \right) && (11) \end{aligned}$$

where $\widehat{\mu} = \mu + \Sigma b_0$ and $\Sigma = (\sigma, 0)$. For convenience, we write μ instead of $\widehat{\mu}$.

We still consider the class $\mathcal{X}$ of semimartingales which have the following :

$$\forall t \in [0,1] \quad X_t = X_0 + \int_0^t \alpha_s ds + \int_0^t \beta_s d\widetilde{B}_s \tag{12}$$

We introduce some notations which will be helpful : $X_t = \begin{pmatrix} X_t^1 \\ X_t^2 \end{pmatrix}$,

$$\forall i \in \{1,2\} \quad dX_t^i = \alpha_t^i dt + \beta_t^{i,\cdot} d\widetilde{B}_t$$

where $\beta^{i,\cdot}$ is the i^{th} row of the matrix β.
From (**??**) and (**??**), we directly have the dynamics of Ψ_t :

$$d\Psi_t = \lambda\left(\alpha + A_0 + A_1 Z_t\right)dt + \lambda\left(\beta + \varepsilon\right)d\widetilde{B}_t \tag{13}$$

Finally, we have to compute the dynamics of the wealth of the insider under F_t. We do it exactly the same way it was done but with little changes. Indeed, the insider can trade on two different assets. His final wealth has the following form :

$$W_{1^+} = B_0 + P_0^T X_0 + \int_0^1 X_{s-}^T dP_s + (\widetilde{v} - S_1^2)X_1^2$$

By using an integration by part, we obtain :

$$W_{1^+} = B_0 + \widetilde{v}^T X_0^2 + \int_0^1 X_s^1 dS_s^1 + \int_0^1 (\widetilde{v} - H)dX_s^2 - \left[S^2, X^2\right]_1$$

As we have done before, we consider the process $\widetilde{W}$ instead of W since it is a continuous process. For more details, we refer to the previous paper. Hence the dynamics of $\widetilde{W}$ under the insider's filtration is :

$$\begin{aligned} d\widetilde{W}_t &= \left(X_t^1 S_t^1 \mu + X_t^1 S_t^1 \Sigma b_1 Z_t + (\widetilde{v} - H)\alpha_2 - \beta^{2,\cdot}(\beta + \varepsilon)^T \lambda \frac{\partial H}{\partial \psi}\right)dt \\ &\quad + \left(X_t^1 S_t^1 \Sigma + (\widetilde{v} - H)\beta^{2,\cdot}\right)d\widetilde{B}_t \end{aligned}$$

4 Characterization of the equilibrium

4.1 General results

We consider the following maximization problem :

$$J(t,r) = \sup_{\alpha,\beta}\left(E\left[U\left(\widetilde{W}_{1^+}\right) \mid R_t = r\right]\right)$$

where $R = \left(X_1, S_1, Z, \Psi, \widetilde{W}\right)$. We have in fact the same proposition as in the previous paper, however the proof is slightly different. The proof is in the Appendix.

Proposition 11 *Among the class $\mathcal{X}$, the semimartingales which have a martingale part cannot satisfy the insider optimality condition.*

Corollary 12 *At the equilibrium, the dynamics of the insider's strategy has to be of the form*

$$dX_t = \alpha_t dt \tag{14}$$

where $\alpha_t \in \Re^N$ is F_t-adapted.

For convenience, let us defined the differential operator $\mathcal{K}$:

$$\begin{aligned}
\mathcal{K}\varphi\left(t,r\right) &= \frac{\partial\varphi}{\partial t} + \frac{\partial\varphi}{\partial s_1}\left(s_1\mu + s_1\Sigma b_1 z\right) + \frac{\partial\varphi}{\partial\psi}\lambda\left(A_0 + A_1 z\right) \\
&+ \frac{\partial\varphi}{\partial\widetilde{w}}\left(x_1 s_1\mu + x_1 s_1\Sigma b_1 z\right) + \frac{\partial\varphi}{\partial z}\left(A_0 + A_1 z\right) \\
&+ \frac{1}{2}s_1^2\Sigma\Sigma^T\frac{\partial^2\varphi}{\partial s_1^2} + \frac{1}{2}tr\left(\lambda\varepsilon\varepsilon^T\lambda\frac{\partial^2\varphi}{\partial\psi^2}\right) \\
&+ tr\left(x_1^2 s_1^2\Sigma\Sigma^T\frac{\partial^2\varphi}{\partial\widetilde{w}^2}\right) + tr\left(s_1\Sigma\varepsilon^T\lambda\frac{\partial^2\varphi}{\partial s_1\partial\psi}\right) + \frac{1}{2}tr\left(\varepsilon\varepsilon^T\frac{\partial^2\varphi}{\partial z^2}\right) \\
&+ tr\left(s_1\Sigma\varepsilon^T\frac{\partial^2\varphi}{\partial s_1\partial z}\right) + tr\left(\lambda\varepsilon\varepsilon^T\frac{\partial^2\varphi}{\partial\psi\partial z}\right) + tr\left(\varepsilon x_1 s_1\Sigma^T\frac{\partial^2\varphi}{\partial z\partial\widetilde{w}}\right) \\
&+ tr\left(x_1 s_1^2\Sigma\Sigma^T\frac{\partial^2\varphi}{\partial s_1\partial\widetilde{w}}\right) + tr\left(\lambda\varepsilon x_1 s_1\Sigma^T\frac{\partial^2\varphi}{\partial\psi\partial\widetilde{w}}\right)
\end{aligned}$$

A direct consequence of the corollary is that we consider in the rest of the paper insider's strategies which have the form given by (**??**). As we see in the proof of this proposition, when (H,λ,J) satisfies the Bellman inequation, we have some necessary conditions for the existence of a solution :

$$\left\{\begin{array}{l}
\frac{\partial J}{\partial\psi_2}\lambda_2 + \frac{\partial J}{\partial\widetilde{w}}\left(\widetilde{v} - H\right) = 0 \\
\frac{\partial J}{\partial\psi_1}\lambda_1 + \frac{\partial J}{\partial x_1} = 0 \\
\lambda_1^2\frac{\partial^2 J}{\partial\psi_1^2} + \frac{\partial^2 J}{\partial x_1^2} \leq 0 \\
\frac{\partial^2 J}{\partial\psi_2^2} \geq 0 \\
\mathcal{K}J \leq 0
\end{array}\right. \tag{15}$$

We remark that the first, the second and the fifth equations of (**??**) are close to what we have before. In fact, if we find (H,λ,J) such that the fifth one is an equation instead of an inequation, we still have a solution. In fact, we look for a solution when there is equality. Besides, the third and the fourth ones will not directly help us in the resolution of the equilibrium even if they have to be satisfied. Now we give a theorem which give sufficient conditions to have a solution.

Theorem 13 *Let* $(H,\lambda) \in \mathcal{H}$, $X_t^* \in \mathcal{X}$, $J \in C^{1,2}\left([0,1],\mathbb{R}^7\right)$ *and* $X_t^* = \int_0^t \alpha_t^* dt$, *satisfy the conditions*

(i) $\quad \frac{\partial J}{\partial \psi_2}\lambda_2 + (\widetilde{v} - H) = 0$

(ii) $\quad \frac{\partial J}{\partial \psi_1}\lambda_1 + \frac{\partial J}{\partial x_1} = 0$

(iii) $\quad \mathcal{K}J = 0$

(iv) $\quad J(1,s_1,x_1,z,\psi,\widetilde{w}) \geq U(\widetilde{w}) \quad \forall s_1, \forall x_1, \forall z, \forall \psi, \forall \widetilde{w}$
$\quad J(1,s_1,x_1,z,\psi,\widetilde{w}) = U(\widetilde{w})$ *if* $H(1,\psi) = \widetilde{v}$.

(v) $\quad E\left(\alpha_t^* \mid F_t^{\Psi^*}\right) = 0$ *where* $\Psi_t^* = \int_0^t \lambda\left(\alpha_s^* ds + dZ_s\right)$

(vi) $\quad H(1,\Psi_1^*) = \widetilde{v}$

Then (H,λ,α^*) *is an equilibrium.*

The proof of this theorem is very close to the one we gave in [6], so we refer to [6] for more details. This verification theorem is similar to the one we have in complete asymmetric information. In fact, the new framework does not change the structure of the theorem. However, the sufficient conditions seem more complicate. We will see in the next section that the informed trader will fix his demand in such a way that the cumulative demand in the first asset does not influence the price formation of the second one.

4.2 Applications

We are going to study some particular cases. We will assume that the insider is risk neutral on one hand and risk averse with exponential utility on the other hand. In fact, this will permit us to compute equilibrium in assuming particular form for the value function J. In those two cases, we can have an interesting result.

Lemma 14 *If we consider* (H,λ,J,α) *which satisfies the conditions (i) to (iii), then* $H(t,\psi_1,\psi_2) = H(t,\psi_2)$.

This means that we are looking for a price which does not depend on the cumulative demand in S_t^1. This can be explained by the fact that the insider does not want to add noise in the price formation of the second asset

4.2.1 Risk Neutral Insider

We assume that the utility function has the following form : $U(x) = x$. Now we are able to compute the equilibrium using Lipster and Shiryaev theory [7]. In fact, we can say that the insider prefers to split his strategy in two parts, one for each assets without direct correlations. So an optimal strategy could be seen as a classical approach of stochastic control with a Black and Scholes price in one hand, and a Back approach for the second asset. We do not say here that this is the only way to attain the equilibrium since we do not have the unicity of such an equilibrium, however, such a strategy will be optimal, hence will constitute an equilibrium. From now, let us assume that ε is constant over time. We know that the computations are slightly different when ε is not constant, but we refer to Lasserre [6] for more details.

Let us define the following couple (H^*, α^*) :

$$H^*\left(t, Y_t^2\right) = E\left(h\left(\widehat{\Theta}\right)\right) \text{ where } \widehat{\Theta} \sim \mathcal{N}\left(m_0 + \sqrt{\frac{\Sigma_0}{\varepsilon_{2,1}^2 + \varepsilon_{2,2}^2}} Y_t^2, \Sigma_0 (1-t)\right)$$

$$\alpha_t^* = \left(\begin{array}{c} \alpha_t^{1*} \\ \frac{1}{1-t}\sqrt{\frac{\varepsilon_{2,1}^2+\varepsilon_{2,2}^2}{\Sigma_0}}\left(h^{-1}(\widetilde{v}) - m_0 - \sqrt{\frac{\Sigma_0}{\varepsilon_{2,1}^2+\varepsilon_{2,2}^2}} Y_t^2\right) \end{array}\right)$$

where α_t^{1*} is F_t-adapted and satisfies $E\left(\alpha_t^{1*} \mid F_t^{\Psi^*}\right) = 0$. Such a process α_t^{1*} exists. For example, $\alpha_t^{1*} = 0$ works. Let us define the vector λ^*

$$\lambda^* = (1,1)^T$$

Proposition 15 *$(H^*, \lambda^*, \alpha^*)$ is an equilibrium. Moreover, we have the following relation :*

$$H(1,y) = h\left(m_0 + \sqrt{\frac{\Sigma_0}{\varepsilon_{2,1}^2 + \varepsilon_{2,2}^2}} y\right)$$

Example 16 We suppose that h has the following form : $h(x) = e^x$, which means that $\widetilde{v}$ has a lognormal distribution. Then the price is :

$$H^*\left(t, Y_t^2\right) = e^{m_0 + \sqrt{\frac{\Sigma_0}{\varepsilon_{2,1}^2+\varepsilon_{2,2}^2}} Y_t^2 + \frac{1}{2}\Sigma_0(1-t)}$$

This is the form of a classical Black and Scholes' price.

4.2.2 Risk Averse Insider

We now assume that the utility function has the following form : $U(x) = -e^{-\eta x}$. From lemma 4.5, we can say that the insider still prefers to split his strategy on the two assets. But, it need more computations in this case to show that the price remains the same as in the total asymmetric information case. Let us focus on the equation satisfied by H if the conditions of theorem 4.3 are satisfied.

Lemma 17 *If conditions (i), (ii) and (iii) of theorem 4.3 are satisfied then (H, λ, j) is a solution of :*

$$\begin{aligned} 0 \quad = \quad & \eta(\widetilde{v} - H)\left(\frac{\lambda_2'}{\lambda_2^2} - \eta\left(\varepsilon_{1,2}^2 + \varepsilon_{2,2}^2\right)\frac{\partial H}{\partial \psi_2}\right) \\ & -\frac{\eta}{\lambda_2}\left(\frac{\partial H}{\partial t} + \frac{1}{2}tr\left(\lambda\varepsilon\varepsilon^T\lambda\frac{\partial^2 H}{\partial \psi^2}\right)\right) + \eta\varepsilon_{1,2}(\varepsilon_{1,1} + \varepsilon_{2,2})\frac{\partial j}{\partial x_1}\frac{\partial H}{\partial \psi_2} \\ & -\eta\sigma s_1\varepsilon_{2,1}\frac{\partial H}{\partial \psi_2}\left(\frac{\partial j}{\partial s_1} - x_1\eta\right) - tr\left(\lambda\varepsilon\varepsilon^T\frac{\partial j}{\partial z}\frac{\eta}{\lambda_2}\frac{\partial H}{\partial \psi}\right) \end{aligned} \tag{16}$$

Our goal is to solve such an equation. But as we saw in [6], the price can not depend on the signal $\widetilde{v}$, since the market maker does not observe the private signal, hence if H satisfies such an equation, we will necessarily have

$$\frac{\lambda_2'}{\lambda_2^2} - \eta\left(\varepsilon_{1,2}^2 + \varepsilon_{2,2}^2\right)\frac{\partial H}{\partial \psi_2} = 0$$

hence, as λ only depends on t, differentiating with respect to ψ_2 we get

$$\frac{\partial^2 H}{\partial \psi_2^2} = 0 \tag{17}$$

Now using (**??**), if we differentiate (**??**) with respect to ψ_2, we obtain :

$$-\frac{\eta}{\lambda_2}\frac{\partial^2 H}{\partial t \partial \psi_2} = 0$$

So, we can conclude that $\frac{1}{\eta\left(\varepsilon_{1,2}^2+\varepsilon_{2,2}^2\right)}\frac{\lambda_2'}{\lambda_2^2} = q$ and $H\left(t, \psi_2\right) = P_0 + q\psi_2$, where q is a constant.

Remark 18 The function H is linear on ψ_2. As in [6], we can say that there is no equilibrium if the signal $\widetilde{v}$ is not Gaussian. In the following, we assume that $\widetilde{v}$ is Gaussian.

Let us define H^*, λ^* and X^* as follows :

$$\lambda^*\left(t\right) = \left(1, \frac{\lambda_2^*\left(1\right)}{\eta\left(\varepsilon_{1,2}^2 + \varepsilon_{2,2}^2\right)\lambda_2^*\left(1\right)\left(1-t\right) + 1}\right)^T$$

where $\lambda_2^*\left(1\right) = \frac{1}{2\sqrt{\varepsilon_{1,2}^2+\varepsilon_{2,2}^2}}\left(\eta\sqrt{\varepsilon_{1,2}^2+\varepsilon_{2,2}^2}\Sigma_0 + \sqrt{\left(\eta\Sigma_0\right)^2\left(\varepsilon_{1,2}^2+\varepsilon_{2,2}^2\right) + 4\Sigma_0}\right)$

$$H\left(t, \Psi_t^{2*}\right) = m_0 + \Psi_t^{2*}$$

where $\Psi_t^{2*} = \int_0^t \lambda_2^*\left(t\right) dY_t^{2*}$ and $Y_t^{2*} = X_t^{2*} + Z_t^2$

$$X_t^* = \left(\frac{1}{\eta\sigma^2 S_t^1}\left(\varphi\left(t\right) + \chi\left(t\right) Z_t\right), \int_0^t \frac{\widetilde{v} - m_0 - \Psi_s^{2*}}{\lambda_2^*\left(1\right)\left(1-s\right)} ds\right)$$

where

$$\varphi\left(t\right) = \mu\left(1-\delta-\delta t\right) - \frac{\mu}{\sigma^2}\theta P\begin{pmatrix} \frac{1}{\delta}\left(e^{\delta(t-1)}-1\right) + (t-1) & 0 \\ 0 & \frac{\delta}{2}\left(1-t\right)^2 \end{pmatrix} P^{-1}\varepsilon\Sigma^T$$

$$\chi\left(t\right) = \sigma\theta - \sigma\theta P\begin{pmatrix} \left(e^{\delta(t-1)}-1\right) & 0 \\ 0 & \delta\left(1-t\right) \end{pmatrix} P^{-1}$$

Proposition 19 $\left(H^*, \lambda^*, X^*\right)$ *is an equilibrium.*

Remark 20 Concerning the second asset, the conclusion is exactly the same as in the risk neutral case. However, we can see the influence of the second asset on the strategy on the first asset. His strategy does not consist on the usual strategy of this kind of portfolio optimization.

5 Conclusion

We show the existence of an equilibrium when the insider just has partial information on the assets. Our results can be extend to a multivariate model where there are d assets without private information and N assets with. For more details about the technical methods, we refer to Lasserre [6]. In this paper, we suppose that the noise trader's demand is a Brownian motion, and not an Ornstein Ulhenbeck process as in the previous paper. We take this assumption to simplify the computations, the equilibrium should still exist in such a case.

Appendix

Proof of Proposition ??

The state variables are X^1, S^1, Z, Ψ and $\widetilde{W}$. If we note $R = \left(X^1, S^1, Z_t, \Psi, \widetilde{W}\right)^T$, we know that the function J satisfies the Jacobi Bellman inequation :

$$\sup_{\alpha,\beta}\left(\frac{\partial J}{\partial t} + L_R J\right) \leq 0$$

Their dynamics are as follows:

$$\begin{aligned}
dX_t^1 &= \alpha_1 dt + \beta^{1,\cdot} dB_t \\
dS_t^1 &= = S_t^1 \left((\mu + \Sigma b_1 Z_t)\, dt + \Sigma d\widetilde{B}t \right) \\
d\Psi_t &= \lambda (\alpha + A_0 + A_1 Z_t)\, dt + \lambda (\beta + \varepsilon)\, d\widetilde{B}_t \\
d\widetilde{W}_t &= \left(X_t^1 S_t^1 \mu + X_t^1 S_t^1 A_1 Z_t + (\widetilde{v} - H)\, \alpha_2 - \beta^{2,\cdot} (\beta + \varepsilon)^T \lambda \frac{\partial H}{\partial \psi} \right) \\
&\quad + \left(X_t^1 S_t^1 \Sigma + (\widetilde{v} - H)\, \beta^{2,\cdot} \right) dB_t
\end{aligned}$$

with $\Sigma = (\sigma, 0) \in M_{1,2}(R)$. We need to compute the expression $\frac{\partial J}{\partial t} + L_R J$:

$$
\begin{aligned}
\frac{\partial J}{\partial t}+L_R J &= \frac{\partial J}{\partial t}+\frac{\partial J}{\partial x_1}\alpha_1+\frac{\partial J}{\partial s_1}\left(s_1\mu+s_1\Sigma b_1 z\right)+\frac{\partial J}{\partial \psi}\lambda\left(\alpha+A_0+A_1 z\right) \\
&+\frac{\partial J}{\partial \widetilde{w}}\left(x_1 s_1\mu+x_1 s_1\Sigma b_1 z+\left(\widetilde{v}-H\right)\alpha_2-\beta^{2,\cdot}\left(\beta+\varepsilon\right)^T\lambda\frac{\partial H}{\partial \psi}\right) \\
&+\frac{\partial J}{\partial z}\left(A_0+A_1 z\right)+\frac{1}{2}tr\left(\varepsilon\varepsilon^T\frac{\partial^2 J}{\partial z^2}\right)+\frac{1}{2}\beta^{1,\cdot}\left(\beta^{1,\cdot}\right)^T\frac{\partial^2 J}{\partial x_1^2} \\
&+\frac{1}{2}s_1^2\Sigma\Sigma^T\frac{\partial^2 J}{\partial s_1^2}+\frac{1}{2}tr\left(\lambda\left(\beta+\varepsilon\right)\left(\beta+\varepsilon\right)^T\lambda\frac{\partial^2 J}{\partial \psi^2}\right) \\
&+\frac{1}{2}tr\left(\left(x_1 s_1\Sigma+\left(\widetilde{v}-H\right)\beta^{2,\cdot}\right)\left(x_1 s_1\Sigma+\left(\widetilde{v}-H\right)\beta^{2,\cdot}\right)^T\frac{\partial^2 J}{\partial \widetilde{w}^2}\right) \\
&+tr\left(s_1\Sigma\varepsilon^T\frac{\partial^2 J}{\partial s_1\partial z}\right)+tr\left(\beta^{1,\cdot}s_1\Sigma^T\frac{\partial^2 J}{\partial x_1\partial s_1}\right) \\
&+tr\left(\beta^{1,\cdot}\lambda\left(\beta+\varepsilon\right)\frac{\partial^2 J}{\partial x_1\partial \psi}\right)+tr\left(\beta^{1,\cdot}\varepsilon^T\frac{\partial^2 J}{\partial x_1\partial z}\right) \\
&+tr\left(\beta^{1,\cdot}\left(x_1 s_1\Sigma+\left(\widetilde{v}-H\right)\beta^{2,\cdot}\right)\frac{\partial^2 J}{\partial x_1\partial \widetilde{w}}\right) \\
&+tr\left(s_1\Sigma\left(\beta+\varepsilon\right)^T\lambda\frac{\partial^2 J}{\partial s_1\partial \psi}\right)+tr\left(\lambda\left(\beta+\varepsilon\right)\varepsilon^T\frac{\partial^2 J}{\partial \psi\partial z}\right) \\
&+tr\left(\varepsilon\left(x_1 s_1\Sigma+\left(\widetilde{v}-H\right)\beta^{2,\cdot}\right)^T\frac{\partial^2 J}{\partial z\partial \widetilde{w}}\right) \\
&+tr\left(s_1\Sigma\left(x_1 s_1\Sigma+\left(\widetilde{v}-H\right)\beta^{2,\cdot}\right)^T\frac{\partial^2 J}{\partial s_1\partial \widetilde{w}}\right) \\
&+tr\left(\lambda\left(\beta+\varepsilon\right)\left(x_1 s_1\Sigma+\left(\widetilde{v}-H\right)\beta^{2,\cdot}\right)^T\frac{\partial^2 J}{\partial \psi\partial \widetilde{w}}\right)
\end{aligned}
$$

For convenience, we show the announced result in a particular case, for more details, we refer to Lasserre [6]. We assume that the insider is risk neutral. We are going to do something classical in this kind of problem, which consists to assume that the function J has the following form when the agent is risk neutral, $J\left(t,x_1,s_1,\psi,\widetilde{w}\right)=\widetilde{w}+\widehat{J}\left(t,x_1,s_1,\psi\right)$.
Hence, $\frac{\partial J}{\partial \widetilde{w}}=1$ and $\frac{\partial^2 J}{\partial \widetilde{w}^2}=\frac{\partial^2 J}{\partial x_1\partial \widetilde{w}}=\frac{\partial^2 J}{\partial s_1\partial \widetilde{w}}=\frac{\partial^2 J}{\partial \psi\partial \widetilde{w}}=0$. So we can write

$$
\begin{aligned}
\frac{\partial J}{\partial t} + L_R J &= \frac{\partial \widehat{J}}{\partial t} + \frac{\partial \widehat{J}}{\partial x_1}\alpha_1 + \frac{\partial \widehat{J}}{\partial s_1}\left(s_1\mu + s_1 b_1 z\right) + \frac{\partial \widehat{J}}{\partial \psi}\lambda\left(\alpha + A_0 + A_1 z\right) \\
&+ \frac{\partial \widehat{J}}{\partial z}\left(A_0 + A_1 z\right) + \left(x_1 s_1 \mu + \left(\widetilde{v} - H\right)\alpha_2 - \beta^{2,\cdot}\left(\beta + \varepsilon\right)^T \lambda \frac{\partial H}{\partial \psi}\right) \\
&+ \frac{1}{2} tr\left(\varepsilon\varepsilon^T \frac{\partial^2 J}{\partial z^2}\right) + \frac{1}{2}\beta^{1,\cdot}\left(\beta^{1,\cdot}\right)^T \frac{\partial^2 \widehat{J}}{\partial x_1^2} + \frac{1}{2} s_1^2 \Sigma\Sigma^T \frac{\partial^2 \widehat{J}}{\partial s_1^2} \\
&+ \frac{1}{2} tr\left(\lambda\left(\beta + \varepsilon\right)\left(\beta + \varepsilon\right)^T \lambda \frac{\partial^2 \widehat{J}}{\partial \psi^2}\right) + tr\left(\beta^{1,\cdot} s_1 \Sigma^T \frac{\partial^2 \widehat{J}}{\partial x_1 \partial s_1}\right) \\
&+ tr\left(\lambda\left(\beta + \varepsilon\right)\beta^{1,\cdot} \frac{\partial^2 \widehat{J}}{\partial x_1 \partial \psi}\right) + tr\left(s_1 \Sigma \varepsilon^T \frac{\partial^2 J}{\partial s_1 \partial z}\right) \\
&+ tr\left(s_1 \Sigma\left(\beta + \varepsilon\right)^T \lambda \frac{\partial^2 \widehat{J}}{\partial s_1 \partial \psi}\right) + tr\left(\beta^{1,\cdot}\varepsilon^T \frac{\partial^2 J}{\partial x_1 \partial z}\right) \\
&+ tr\left(\lambda\left(\beta + \varepsilon\right)\varepsilon^T \frac{\partial^2 J}{\partial \psi \partial z}\right)
\end{aligned}
$$

We find the maximum with respect to α_2 in using the argument that this maximum could not be infinite so necessarily :

$$
\frac{\partial \widehat{J}}{\partial \psi_2}\lambda_2 + \left(\widetilde{v} - H\right) = 0 \tag{A.18}
$$

With the same argument, we can say that if the maximum in α_1 is finite, then

$$
\frac{\partial \widehat{J}}{\partial \psi_1}\lambda_1 + \frac{\partial \widehat{J}}{\partial x_1} = 0 \tag{A.19}
$$

The main part of the proof is to show that necessarily $\beta = 0$, so we have to find the maximum of the following quantity

$$
\begin{aligned}
\Gamma = &-\beta^{2,\cdot}\left(\beta + \varepsilon\right)^T \lambda \frac{\partial H}{\partial \psi} + \frac{1}{2}\beta^{1,\cdot}\left(\beta^{1,\cdot}\right)^T \frac{\partial^2 \widehat{J}}{\partial x_1^2} + \frac{1}{2} tr \lambda\beta\beta^T \lambda \frac{\partial^2 \widehat{J}}{\partial \psi^2} \\
&+ tr\left(\lambda\beta\varepsilon^T \lambda \frac{\partial^2 \widehat{J}}{\partial \psi^2}\right) + tr\left(\beta^{1,\cdot} s_1 \Sigma^T \frac{\partial^2 \widehat{J}}{\partial x_1 \partial s_1}\right) + \lambda\left(\beta + \varepsilon\right)\beta^{1,\cdot} \frac{\partial^2 \widehat{J}}{\partial x_1 \partial \psi} \\
&+ tr\left(s_1 \Sigma\beta^T \lambda \frac{\partial^2 \widehat{J}}{\partial s_1 \partial \psi}\right) + tr\left(\beta^{1,\cdot}\varepsilon^T \frac{\partial^2 J}{\partial x_1 \partial z}\right) + tr\left(\lambda\beta\varepsilon^T \frac{\partial^2 J}{\partial \psi \partial z}\right)
\end{aligned}
$$

Using (**??**) and (**??**), in differentiating them with respect to ψ, we see that some terms of Π collapse, so we get

$$\Pi = -\beta^{2,\cdot}\beta^T\lambda\frac{\partial H}{\partial\psi} + \frac{1}{2}\beta^{1,\cdot}\left(\beta^{1,\cdot}\right)^T\frac{\partial^2\widehat{J}}{\partial x_1^2} + \frac{1}{2}tr\left(\lambda\beta\beta^T\lambda\frac{\partial^2\widehat{J}}{\partial\psi^2}\right)$$
$$+tr\left(\beta^{1,\cdot}s_1\Sigma^T\frac{\partial^2\widehat{J}}{\partial x_1\partial s_1}\right) + tr\left(s_1\Sigma\beta^T\lambda\frac{\partial^2\widehat{J}}{\partial s_1\partial\psi}\right)$$

Then, differentiating (**??**) with respect to s_1, we obtain :

$$\Pi = -\beta^{2,\cdot}\beta^T\lambda\frac{\partial H}{\partial\psi} + \frac{1}{2}\beta^{1,\cdot}\left(\beta^{1,\cdot}\right)^T\frac{\partial^2\widehat{J}}{\partial x_1^2} + \frac{1}{2}tr\left(\lambda\beta\beta^T\lambda\frac{\partial^2\widehat{J}}{\partial\psi^2}\right) + s_1\sigma\beta^{2,1}\lambda_2\frac{\partial^2\widehat{J}}{\partial s_1\partial\psi_2}$$

Finally, recalling that H only depends on t and Ψ, when we differentiate (**??**) with respect to s_1, we get $\frac{\partial^2\widehat{J}}{\partial s_1\partial\psi_2} = 0$. Hence, the expression we have to maximize is now :

$$\Pi = -\beta^{2,\cdot}\beta^T\lambda\frac{\partial H}{\partial\psi} + \frac{1}{2}\beta^{1,\cdot}\left(\beta^{1,\cdot}\right)^T\frac{\partial^2\widehat{J}}{\partial x_1^2} + \frac{1}{2}tr\left(\lambda\beta\beta^T\lambda\frac{\partial^2\widehat{J}}{\partial\psi^2}\right)$$

We switch the partial derivatives of H by those of J using (**??**) and (**??**), and finally, we get a quadratic form in the coefficients of the matrix β :

$$\Pi = \frac{1}{2}\left(\lambda_1^2\frac{\partial^2\widehat{J}}{\partial\psi_1^2} + \frac{\partial^2\widehat{J}}{\partial x_1^2}\right)\left(\beta_{1,1}^2 + \beta_{1,2}^2\right) - \frac{1}{2}\lambda_2^2\frac{\partial^2\widehat{J}}{\partial\psi_2^2}\left(\beta_{2,1}^2 + \beta_{2,2}^2\right)$$

It is quite clear easy that we have two conditions for the existence of a maximum of the expression Π which are :

$$\begin{aligned}\lambda_1^2\frac{\partial^2\widehat{J}}{\partial\psi_1^2} + \frac{\partial^2\widehat{J}}{\partial x_1^2} &\leq 0 \\ \frac{\partial^2\widehat{J}}{\partial\psi_2^2} &\geq 0\end{aligned} \qquad \text{(A.20)}$$

In the case where those inequality are satisfied, the maximum of this expression is zero and this occurs if and only if :

$$\beta_{1,1} = \beta_{1,2} = \beta_{2,1} = \beta_{2,2} = 0$$

which means : $\beta = 0$. In any other case, the expression Π has a infinite maximum which means that no equilibrium could exist.
Proof of Lemma ?? Let us assume that $(H, \lambda, J, \alpha^*)$ satisfies the conditions (i), (ii) and (iii).

We first show the result in the risk neutral case, hence $U(x) = x$. We assume that the value function J has the following form :

$$J(t, x_1, s_1, z, \psi, \widetilde{w}) = \widetilde{w} + \widehat{J}(t, x_1, s_1, z, \psi)$$

We have

$$\frac{\partial \widehat{J}}{\partial \psi_2}\lambda_2 = H - \widetilde{v}$$

hence

$$\widehat{J}(t,R) = \int_0^{\psi_2} \frac{H(1,\psi_1,y_2) - \widetilde{v}}{\lambda_2} dy_2 + A(t,\psi_1,z,s_1,x_1)$$

But we also know that

$$\lambda_1 \frac{\partial \widehat{J}}{\partial \psi_1} + \frac{\partial \widehat{J}}{\partial x_1} = 0$$

so we have :

$$\lambda_1 \frac{\partial A}{\partial \psi_1} + \lambda_1 \int_0^{\psi_2} \frac{1}{\lambda_2} \frac{\partial H}{\partial \psi_1}(t,\psi_1,y)\,dy + \frac{\partial A}{\partial x_1} = 0$$

$$\lambda_1 \frac{\partial A}{\partial \psi_1}(t,\psi_1,z,s_1,x_1) + \frac{\partial A}{\partial x_1}(t,\psi_1,z,s_1,x_1) = -\lambda_1 \int_0^{\psi_2} \frac{1}{\lambda_2} \frac{\partial H}{\partial \psi_1}(t,\psi_1,y)\,dy$$

We remark that the left hand side does not depend on ψ_2, hence we take $\psi_2 = 0$ in the right hand side :

$$\forall (t,\psi_1,s_1,x_1) \quad \lambda_1 \frac{\partial A}{\partial \psi_1}(t,\psi_1,s_1,x_1) + \frac{\partial A}{\partial x_1}(t,\psi_1,s_1,x_1) = 0$$

So finally we have :

$$\forall (t,\psi_1,\psi_2) \quad \int_0^{\psi_2} \frac{1}{\lambda_2} \frac{\partial H}{\partial \psi_1}(t,\psi_1,y)\,dy = 0$$

which means that necessarily : $\frac{\partial H}{\partial \psi_1}(t,\psi_1,y) = 0$, hence H does not depend on ψ_1. The proof for the risk averse case is slightly the same. Here, the utility function is $U(x) = -e^{-\eta x}$ where $\eta > 0$. We assume then that the value function has the following form :

$$J(t,x_1,s_1,z,\psi,\widetilde{w}) = -e^{-\eta \widetilde{w} + j(t,x_1,s_1,z,\psi)}$$

Hence, we can compute the partial derivatives of J

$$\begin{aligned} \frac{\partial J}{\partial u} &= \frac{\partial j}{\partial u} J \text{ for } u \in \{x_1,s_1,z,t,\psi\} \\ \frac{\partial J}{\partial \widetilde{w}} &= -\eta J \end{aligned}$$

So conditions (i) and (ii) become :

$$\begin{aligned} \frac{\partial j}{\partial \psi_2}\lambda_2 &= \eta(\widetilde{v} - H) \\ \lambda_1 \frac{\partial j}{\partial \psi_1} &= -\frac{\partial j}{\partial x_1} \end{aligned}$$

We obtain exactly the same kind of equations as in the risk neutral case, hence the conclusion is the same.
Proof of Proposition ?? This is a direct application of the fact that the insider can choose to trade S_t^1 and S_t^2 separately. On one hand, the strategy of a risk neutral agent on an asset such as S_t^1 can be everything. On the other hand, we compute the price and the strategy α^2 as it was done in Lasserre with the following system

$$\begin{cases} d\Theta = 0 \\ dY_t^2 = \alpha_t^2 dt + \varepsilon^{2,\cdot} d\widetilde{B}_t \end{cases}$$

and we look for a strategy which has the form : $\alpha_t^2 = \delta_1\left(t, Y^2\right) + \delta_2\left(t, Y^2\right)\Theta$.
Proof of Lemma ?? We are looking for a particular form of the value function. Actually, let us suppose that :

$$\begin{aligned} J\left(t, x_1, s_1, \psi, \widetilde{w}\right) &= -e^{-\eta \widetilde{w} + j(t, x_1, s_1, \psi)} \\ j\left(1, x_1, s_1, \psi\right) &= 0 \end{aligned}$$

Hence, we can rewrite the derivatives.
$\frac{\partial J}{\partial \widetilde{w}} = -\eta J$, $\frac{\partial^2 J}{\partial \widetilde{w}^2} = \eta^2 J$, $\frac{\partial J}{\partial y} = \frac{\partial j}{\partial y} J$ when $y = t, x_1, s_1, z, \psi$,
$\frac{\partial^2 J}{\partial y \partial u} = \left(\frac{\partial^2 j}{\partial y \partial u} + \frac{\partial j}{\partial y}\frac{\partial j}{\partial u}\right) J$ when $y = x_1, s_1, z, \psi$ and $u = x_1, s_1, z, \psi$. The first equation becomes : $\frac{\partial j}{\partial \psi_2}\lambda_2 - \eta\left(\widetilde{v} - H\right) = 0$. The second equation remains the same : $\frac{\partial j}{\partial \psi_1}\lambda_1 + \frac{\partial j}{\partial x_1} = 0$
The third equation of the previous system becomes :

$$\begin{aligned} 0 = {} & \frac{\partial j}{\partial t} + \frac{\partial j}{\partial s_1}\left(s_1\mu + s_1 b_1 z\right) - \eta\left(x_1 s_1 \mu + x_1 s_1 \Sigma b_1 z\right) \\ & + \frac{\partial j}{\partial \psi}\lambda\left(A_0 + A_1 z\right) + \frac{1}{2}s_1^2\sigma^2\left(\frac{\partial^2 j}{\partial s_1^2} + \left(\frac{\partial j}{\partial s_1}\right)^2\right) \\ & + \frac{\partial j}{\partial z}\left(A_0 + A_1 z\right) + \frac{1}{2}tr\left(\lambda\varepsilon\varepsilon^T\lambda\left(\frac{\partial^2 j}{\partial \psi^2} + \frac{\partial j}{\partial \psi}\left(\frac{\partial j}{\partial \psi}\right)^T\right)\right) \\ & + \frac{1}{2}tr\left(\varepsilon\varepsilon^T\left(\frac{\partial^2 j}{\partial z^2} + \frac{\partial j}{\partial z}\left(\frac{\partial j}{\partial z}\right)^T\right)\right) + \frac{1}{2}x_1^2 s_1^2\sigma^2\eta^2 \\ & + tr\left(s_1\Sigma\varepsilon^T\lambda\left(\frac{\partial^2 j}{\partial s_1 \partial \psi} + \frac{\partial j}{\partial s_1}\frac{\partial j}{\partial \psi}\right)\right) - \eta x_1 s_1^2\sigma^2\frac{\partial j}{\partial s_1} \\ & - tr\left(\eta\lambda\varepsilon x_1 s_1 \Sigma^T\frac{\partial j}{\partial \psi}\right) + tr\left(s_1\Sigma\varepsilon^T\left(\frac{\partial^2 j}{\partial s_1 \partial z} + \frac{\partial j}{\partial s_1}\frac{\partial j}{\partial z}\right)\right) \\ & + tr\left(\lambda\varepsilon\varepsilon^T\left(\frac{\partial^2 j}{\partial z \partial \psi} + \frac{\partial j}{\partial z}\frac{\partial j}{\partial \psi}\right)\right) - tr\left(\eta x_1 s_1 \varepsilon^T\frac{\partial j}{\partial z}\right) \end{aligned} \tag{A.21}$$

Let us differentiate the first equation two times with respect to ψ

$$\lambda_2\frac{\partial^3 j}{\partial \psi_2 \partial \psi^2} = -\eta\frac{\partial^2 H}{\partial \psi^2}$$

Now, we remark that $\frac{\partial^2 j}{\partial s_1 \partial \psi_2} = \frac{\partial^2 j}{\partial z \partial \psi_2} = 0$, since H does not depend on s_1 and on z, and we differentiate (**??**) with respect to ψ_2, hence we get, by substituting $\frac{\partial^3 j}{\partial \psi_2 \partial \psi^2}$:

$$\begin{aligned} 0 &= \frac{\partial^2 j}{\partial t \partial \psi_2} + \frac{1}{2} tr\left(\lambda \varepsilon \varepsilon^T \lambda \left(-\frac{\eta}{\lambda_2}\frac{\partial^2 H}{\partial \psi^2} + 2\frac{\partial^2 j}{\partial \psi \partial \psi_2}\left(\frac{\partial j}{\partial \psi}\right)^T\right)\right) \\ &+ s_1\left(tr\left(\Sigma \varepsilon^T \lambda \frac{\partial j}{\partial s_1}\frac{\partial^2 j}{\partial \psi \partial \psi_2}\right) - tr\left(\eta \lambda \varepsilon x_1 \Sigma^T \frac{\partial^2 j}{\partial \psi \partial \psi_2}\right)\right) \\ &+ tr\left(\lambda \varepsilon \varepsilon^T \frac{\partial j}{\partial z}\frac{\partial^2 j}{\partial \psi \partial \psi_2}\right) \end{aligned} \tag{A.22}$$

Then, we differentiate the first equation with respect to t :

$$\begin{aligned} \lambda_2 \frac{\partial^2 j}{\partial \psi_2 \partial t} + \lambda_2' \frac{\partial j}{\partial \psi_2} &= -\eta \frac{\partial H}{\partial t} \\ \lambda_2 \frac{\partial^2 j}{\partial \psi_2 \partial t} &= -\eta \frac{\partial H}{\partial t} + \frac{\lambda_2'}{\lambda_2}\eta\left(\widetilde{v} - H\right) \end{aligned}$$

So we plug this in (**??**), and we get :

$$\begin{aligned} 0 &= -\frac{\eta}{\lambda_2}\frac{\partial H}{\partial t} + \frac{\lambda_2'}{\lambda_2^2}\eta\left(\widetilde{v} - H\right) + \frac{1}{2} tr\left(\lambda \varepsilon \varepsilon^T \lambda \left(-\frac{\eta}{\lambda_2}\frac{\partial^2 H}{\partial \psi^2} + 2\frac{\partial^2 j}{\partial \psi \partial \psi_2}\left(\frac{\partial j}{\partial \psi}\right)^T\right)\right) \\ &+ s_1 tr\left(\Sigma \varepsilon^T \lambda \frac{\partial^2 j}{\partial \psi \partial \psi_2}\right)\left(\frac{\partial j}{\partial s_1} - x_1 \eta\right) + tr\left(\lambda \varepsilon \varepsilon^T \frac{\partial j}{\partial z}\frac{\partial^2 j}{\partial \psi \partial \psi_2}\right) \end{aligned}$$

Now, our goal is to compute $\frac{\partial^2 j}{\partial \psi \partial \psi_2}\left(\frac{\partial j}{\partial \psi}\right)^T$. But we know that $\frac{\partial^2 j}{\partial \psi \partial \psi_2} = -\frac{\eta}{\lambda_2}\frac{\partial H}{\partial \psi} = -\frac{\eta}{\lambda_2}\begin{pmatrix} 0 \\ \frac{\partial H}{\partial \psi_2} \end{pmatrix}$. However, we also know that $\frac{\partial j}{\partial \psi} = \begin{pmatrix} -\frac{1}{\lambda_1}\frac{\partial j}{\partial x_1} \\ \frac{\eta}{\lambda_2}\left(\widetilde{v} - H\right) \end{pmatrix}$, by looking at the two first equations of the system. Hence,

$$\frac{\partial^2 j}{\partial \psi \partial \psi_2}\left(\frac{\partial j}{\partial \psi}\right)^T = \begin{pmatrix} 0 & 0 \\ \frac{\eta}{\lambda_1 \lambda_2}\frac{\partial j}{\partial x_1}\frac{\partial H}{\partial \psi_2} & -\frac{\eta^2}{\lambda_2^2}\left(\widetilde{v} - H\right)\frac{\partial H}{\partial \psi_2} \end{pmatrix}$$

Hence the equation becomes :

$$\begin{aligned} 0 &= -\frac{\eta}{\lambda_2}\frac{\partial H}{\partial t} + \frac{\lambda_2'}{\lambda_2^2}\eta\left(\widetilde{v} - H\right) - \frac{\eta}{2\lambda_2} tr\left(\lambda \varepsilon \varepsilon^T \lambda \frac{\partial^2 H}{\partial \psi^2}\right) \\ &+ \eta \varepsilon_{1,2}\left(\varepsilon_{1,1} + \varepsilon_{2,2}\right)\frac{\partial j}{\partial x_1}\frac{\partial H}{\partial \psi_2} - \eta^2\left(\varepsilon_{1,2}^2 + \varepsilon_{2,2}^2\right)\left(\widetilde{v} - H\right)\frac{\partial H}{\partial \psi_2} \\ &+ s_1 tr\left(\Sigma \varepsilon^T \lambda \frac{\partial^2 j}{\partial \psi \partial \psi_2}\right)\left(\frac{\partial j}{\partial s_1} - x_1 \eta\right) + tr\left(\lambda \varepsilon \varepsilon^T \frac{\partial j}{\partial z}\frac{\partial^2 j}{\partial \psi \partial \psi_2}\right) \end{aligned}$$

So we get

$$\begin{aligned}
0 &= -\frac{\eta}{\lambda_2}\left(\frac{\partial H}{\partial t}+\frac{1}{2}tr\left(\lambda\varepsilon\varepsilon^T\lambda\frac{\partial^2 H}{\partial\psi^2}\right)\right)+\eta\left(\widetilde{v}-H\right)\left(\frac{\lambda_2'}{\lambda_2^2}-\eta\left(\varepsilon_{1,2}^2+\varepsilon_{2,2}^2\right)\frac{\partial H}{\partial\psi_2}\right)\\
&\quad+\eta\varepsilon_{1,2}\left(\varepsilon_{1,1}+\varepsilon_{2,2}\right)\frac{\partial j}{\partial x_1}\frac{\partial H}{\partial\psi_2}-s_1 tr\left(\Sigma\varepsilon^T\lambda\frac{\eta}{\lambda_2}\frac{\partial H}{\partial\psi}\right)\left(\frac{\partial j}{\partial s_1}-x_1\eta\right)\\
&\quad-tr\left(\lambda\varepsilon\varepsilon^T\frac{\partial j}{\partial z}\frac{\eta}{\lambda_2}\frac{\partial H}{\partial\psi}\right) \qquad (A.23)
\end{aligned}$$

Finally, the equation given in the lemma is obtained by substituting Σ by $(\sigma,0)$.
Proof of Proposition ?? To show this proposition, it suffices to remark that the insider prefers to trade the two assets separately. Hence the equilibrium is just two equilibrium put together. On one hand, he trades on a asset with asymmetric information, and we refer to [6]. On the other hand, it is a classical stochastic control problem. The price process has the following dynamics

$$dS_t^1 = S_t^1\left(\left(\mu+\Sigma b_0+\Sigma b_1 Z_t\right)dt+\Sigma d\widetilde{B}_t\right)$$

where μ and Σ are deterministic. For convenience, we defined $\mu\equiv\mu+\Sigma b_0$. So, we need to know the dynamics of Z since it appears as a state variable

$$dZ_t=\varepsilon d\widetilde{B}_t$$

Then the wealth process can be written :

$$dV_t=\pi_t\left(\mu+\Sigma b_1 Z_t\right)dt+\pi_t\sigma dB_t$$

where π is the amount in the first asset. The optimization problem can be solved by a Bellman approach. We consider the function J

$$J\left(t,z,v\right)=\sup_{\pi\in\Pi}\left(E\left(-e^{-\eta V_1}\mid Z_t=z,V_t=v\right)\right)$$

We know that J is solution of the Bellman equation

$$\frac{\partial J}{\partial t}+\frac{1}{2}tr\left(\varepsilon\varepsilon^T\frac{\partial^2 J}{\partial z^2}\right)+\sup_{\pi}\left(\pi_t\left(\mu\Sigma b_1 z\right)\frac{\partial J}{\partial v}+\pi_t\Sigma\varepsilon^T\frac{\partial^2 J}{\partial v\partial z}+\frac{1}{2}\pi_t^2\sigma^2\frac{\partial^2 J}{\partial v^2}\right)=0$$

and the terminal condition is

$$\forall z\forall v\quad J\left(1,z,v\right)=U\left(v\right)=-e^{-\eta v}$$

We assume that the function J has the following form $J\left(t,v\right)=-e^{-\eta v+j(t,z)}$. Hence the Bellman equation becomes

$$\begin{aligned}
0 &= \frac{\partial j}{\partial t}+\frac{1}{2}tr\left(\varepsilon\varepsilon^T\left(\frac{\partial^2 j}{\partial z^2}+\frac{\partial j}{\partial z}\frac{\partial j}{\partial z}^T\right)\right)\\
&\quad+\sup_{\pi}\left(-\eta\pi_t\left(\mu+\Sigma b_1 z+\Sigma\varepsilon^T\frac{\partial j}{\partial z}\right)+\frac{1}{2}\pi_t^2\sigma^2\eta^2\right)
\end{aligned}$$

with terminal condition $\forall z \quad j(1,z)=0$
The maximum is attained for $\pi_t^* = \frac{\mu+\Sigma b_1 z_1+\Sigma\varepsilon^T \frac{\partial j}{\partial z}}{\eta\sigma^2}$. So, we get

$$\frac{\partial j}{\partial t}+\frac{1}{2}tr\left(\varepsilon\varepsilon^T\left(\frac{\partial^2 j}{\partial z^2}+\frac{\partial j}{\partial z}\frac{\partial j}{\partial z}^T\right)\right)-\frac{\left(\mu+\Sigma b_1 z+\Sigma\varepsilon^T\frac{\partial j}{\partial z}\right)^2}{2\sigma^2}=0$$

Hence to have the optimal portfolio, we need to compute j. We remark that we can simplify this expression by developing it :

$$\frac{\partial j}{\partial t}+\frac{1}{2}tr\left(\varepsilon\varepsilon^T\frac{\partial^2 j}{\partial z^2}\right)-\frac{1}{2\sigma^2}\left(\mu+\Sigma b_1 z\right)^2-\frac{1}{\sigma^2}\left(\mu+\Sigma b_1 z\right)\Sigma\varepsilon^T\frac{\partial j}{\partial z}=0$$

Using Feyman Kac formula, we can solved this partial differential equation

$$j(t,z)=E^{\mathbb{Q}}[-\frac{1}{2\sigma^2}\int_t^1\left(\mu+\Sigma b_1 Z_u\right)^2du \mid Z_t=z]$$

where $dZ_u=\varepsilon dB_u^{\mathbb{Q}}-\frac{1}{\sigma^2}\varepsilon\Sigma^T\left(\mu+\Sigma b_1 Z_u\right)$ with $B_u^{\mathbb{Q}}$ a $\mathbb{Q}$ Brownian motion. We remark that under $\mathbb{Q}$, Z is an Ornstein Ulhenbeck process. Hence we are able to compute explicitly the function j . Considering the remark we made, we have that conditioning on $Z_t=z$, we have

$$\forall u>t \quad Z_u=e^{C(t-u)}z-\int_t^u e^{C(s-u)}a\,ds+\int_t^u e^{C(s-u)}\varepsilon dB_s^{\mathbb{Q}}$$

where $C=\begin{pmatrix}\varepsilon_{1,1}b_{1,1}^1 & \varepsilon_{1,1}b_{1,2}^1\\ \varepsilon_{1,2}b_{1,1}^1 & \varepsilon_{1,2}b_{1,2}^1\end{pmatrix}=\frac{1}{\sigma^2}\varepsilon\Sigma^T\Sigma b_1$ and $a=\frac{1}{\sigma^2}\varepsilon\Sigma^T\mu$. We remark that C is singular, so we have two cases to study.
(i) If $\delta=\varepsilon_{1,1}b_{1,1}^1+\varepsilon_{1,2}b_{1,2}^1\neq 0$, then

$$\begin{aligned}C &= \begin{pmatrix}\varepsilon_{1,1} & b_{1,2}^1\\ \varepsilon_{1,2} & -b_{1,1}^1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}b_{1,1}^1 & \varepsilon_{1,2}\\ b_{1,2}^1 & -\varepsilon_{1,1}\end{pmatrix}\\ &= P\begin{pmatrix}\delta & 0\\ 0 & 0\end{pmatrix}P^{-1}\end{aligned}$$

thus, we can compute $e^{C(s-u)}$

$$e^{C(s-u)}=P\begin{pmatrix}e^{\delta(s-u)} & 0\\ 0 & 1\end{pmatrix}P^{-1}$$

(ii) If $\delta=0$, then $C^2=0$, hence $e^{C(s-u)}=I_2+C(s-u)$
We assume that $\delta\neq 0$, which seems the more credible assumptions since ε and b_1 are not correlated.

$$\begin{aligned}\int_t^u e^{C(s-u)}a\,ds &= P\begin{pmatrix}\int_t^u e^{\delta(s-u)}ds & 0\\ 0 & 1\end{pmatrix}P^{-1}a\\ &= P\begin{pmatrix}\frac{1}{\delta}\left(e^{\delta(t-u)}-1\right) & 0\\ 0 & u-t\end{pmatrix}P^{-1}a\end{aligned}$$

hence, we can write
$\forall u \in [t, 1]$

$$E^{\mathbb{Q}}[Z_u \mid Z_t = z] = P\left(\begin{pmatrix} e^{\delta(t-u)} & 0 \\ 0 & 1 \end{pmatrix} P^{-1} z + \begin{pmatrix} \frac{1}{\delta}\left(e^{\delta(t-u)} - 1\right) & 0 \\ 0 & u - t \end{pmatrix} P^{-1} a\right) \tag{A.24}$$

In fact, we only need to know $\frac{\partial j}{\partial z}$ to compute π_t, so we focus on this computation

$$\begin{aligned} \frac{\partial j}{\partial z}(t, z) &= \frac{\partial}{\partial z} E^{\mathbb{Q}}[-\frac{1}{2\sigma^2} \int_t^1 (\mu + \Sigma b_1 Z_u)^2 \, du \mid Z_t = z] \\ &= \frac{\partial}{\partial z}[\int_t^1 E^{\mathbb{Q}}[-\frac{1}{2\sigma^2}(\mu + \Sigma b_1 Z_u)^2 \mid Z_t = z] du] \end{aligned}$$

The last equality comes from the Fubini theorem for a nonpositive function. But, we can show that $E^{\mathbb{Q}}[-\frac{1}{2\sigma^2}(\mu + \Sigma b_1 Z_u)^2 du \mid Z_t = z]$ is a quadratic form with respect to z (it suffices to compute the first moments of a Ornstein Ulhenbeck process). Hence, $(u, z) \to E^{\mathbb{Q}}[-\frac{1}{2\sigma^2}(\mu + \Sigma b_1 Z_u)^2 du \mid Z_t = z]$ is continuous on compact sets of the form $[t, 1] \times K$, hence we can switch the derivative and the integral which leads

$$\frac{\partial j}{\partial z}(t, z) = \int_t^1 \frac{\partial}{\partial z}[E^{\mathbb{Q}}[-\frac{1}{2\sigma^2}(\mu + \Sigma b_1 Z_u)^2 du \mid Z_t = z]] du$$

We are going to show that we can switch the conditional expected value and the derivative. In fact, we need to show that the family $\frac{\partial f}{\partial z}(Z_u) \mid Z_t = z$ is uniformly integrable in some neighborhood of z. We know that $\frac{\partial f}{\partial z}(Z_u) = A + BZ_u$, hence

$$\left\| E\left(\frac{\partial f}{\partial z}(Z_u) \mid Z_t = z'\right) \right\| = \left\| A + BE^{\mathbb{Q}}[Z_u \mid Z_t = z'] \right\| < +\infty \quad \forall z' \in B(z, 1)$$

So, we can switch the derivative and the conditional expectation. By the way, $\frac{\partial f}{\partial z}(z) = -\frac{1}{\sigma}\mu\theta^T - \theta\theta^T z$ where $\theta = b_1^{1,\cdot}$ is the first line of the matrix b_1. Finally we get :

$$\frac{\partial}{\partial z}[E^{\mathbb{Q}}[-\frac{1}{2\sigma^2}(\mu + \Sigma b_1 Z_u)^2 du \mid Z_t = z]] = -\frac{1}{\sigma}\mu\theta^T - \theta\theta^T E^{\mathbb{Q}}[Z_u \mid Z_t = z]$$

by using (**??**), we get

$$\begin{aligned} \frac{\partial}{\partial z}[E^{\mathbb{Q}}[-\frac{1}{2\sigma^2}(\mu + \Sigma b_1 Z_u)^2 du \quad \mid \quad Z_t = z]] &= -\frac{1}{\sigma}\mu\theta^T - \theta\theta^T e^{C(t-u)} z \\ &\quad -\theta\theta^T P \begin{pmatrix} \frac{1}{\delta}\left(e^{\delta(t-u)} - 1\right) & 0 \\ 0 & u - t \end{pmatrix} P^{-1} a \end{aligned}$$

To conclude, we see that

$$\frac{\partial j}{\partial z}(t, z) = A(t) + B(t) z$$

where
$A(t) = -\frac{1}{\sigma}\mu\theta^T(1-t) - \theta\theta^T P \begin{pmatrix} \frac{1}{\delta^2}\left(e^{\delta(t-1)}-1\right) + \frac{1}{\delta}(t-1) & 0 \\ 0 & \frac{1}{2}(1-t)^2 \end{pmatrix} P^{-1}a$
and $B(t) = -\theta\theta^T P \begin{pmatrix} \frac{1}{\delta}\left(e^{\delta(t-1)}-1\right) & 0 \\ 0 & 1-t \end{pmatrix} P^{-1}$.
Now, we are able to compute explicitly $X_t^{1*} = \frac{\pi_t^*}{S_t^1}$

$$X_t^{1*} = \frac{1}{\eta\sigma^2 S_t^1}\left(\varphi(t) + \chi(t) Z_t\right)$$

where
$\varphi(t) = \mu(1-\delta-\delta t) - \frac{\mu}{\sigma^2}\theta P \begin{pmatrix} \frac{1}{\delta}\left(e^{\delta(t-1)}-1\right) + (t-1) & 0 \\ 0 & \frac{\delta}{2}(1-t)^2 \end{pmatrix} P^{-1}\varepsilon\Sigma^T$
$\chi(t) = \sigma\theta - \sigma\theta P \begin{pmatrix} \left(e^{\delta(t-1)}-1\right) & 0 \\ 0 & \delta(1-t) \end{pmatrix} P^{-1}$

References

1. Back, K.: Insider trading in continuous time. Review of Financial Studies, (**5**), (1992)

2. Bhattacharya, U., Reny, P.J.,Spiegel, M.: Destructive Interference in a Imperfectly Competitive Multi-Security Market. Journal of Economic Theory (**65**), 136 - 170, (1995).

3. Caballe, J., Krishnan, M.: Imperfect competition in a multi-security market with risk neutrality. Econometrica (**62**), 695-704, (1994).

4. Cho, K.H.: Equilibre avec asymétrie d'information. PhD thesis, Université Paris 6, january 1997.

5. Kyle, A.S.: Continuous auctions and insider trading. Econometrica, (**53**), (1985).

6. Lasserre, G. : Asymmetric information and imperfect competition in a continuous time multivariate security model. Preprint PMA 667, Laboratoire de Probabilités et Modèles Aléatoires, Université Paris 6 et Paris 7 (2001)

7. Lipster, R.S., Shiryaev, A.N.: Statistics of random processes. volume I and II. Springer-Verlag, (1976).