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Decision Making with Imprecise Probabilistic Information*

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Abstract

We develop an axiomatic approach to decision under objective imprecision. The information is described by a set of priors and a reference prior. We define a notion of imprecision for this informational setting and show that a decision maker who is averse to information imprecision maximizes the minimum expected utility computed with respect to a *subset* of the set of initially given priors. The reduction of this set can be seen as a measure of imprecision aversion. This approach allows us thus allows a lot of flexibility in modelling the decision maker attitude towards imprecision. In contrast, applying Gilboa and Schmeidler (1989) maxmin criterion to the initial set of priors amounts to assuming extreme pessimism.

Key words: decision making under uncertainty, set of priors, objective information.

Résumé

Nous proposons un approche axiomatique de la décision dans l'incertain avec une information objective imprécise. L'information est représentée par un ensemble de croyances, associé à une croyance de référence. Nous définissons une notion d'imprécision dans ce cadre, et montrons qu'un décideur qui présente de l'aversion à l'égard de l'imprécision de l'information maximise le minimum de l'espérance d'utilité, calculé par rapport à un *sous-ensemble* de l'ensemble de croyances dont il dispose. Le degré de réduction de cet ensemble peut être vu comme une mesure de l'aversion à l'égard de l'imprécision de l'information. Cette approche autorise un grand degré de flexibilité dans la modélisation de l'attitude du décideur à l'égard de l'imprécision de l'information, tandis que le critère maxmin de Gilboa and Schmeidler (1989), appliqué à l'ensemble d'information initial revient à supposer un degré de pessimisme extrême.

Key words: décision dans l'incertain, ensemble de croyances, information objective.

1 Introduction

Ever since the contributions of de Finetti (1937) and Savage (1954) the subjectivist approach to statistics and decision theory holds the view that probabilities do not "objectively" exist but lie in the head of decision makers. These probabilities can hence at best be revealed through choice behavior. Interestingly, economics has by and large adopted an opposite attitude and retained von Neumann and Morgenstern (1947) approach, based on the idea that agents choose among risky prospects, that is in situations where "probabilities objectively exist": a wide spread assumption is that agents share the same beliefs (e.g. the principal and the agent in incentive theory) which are thus considered as reflecting "objective information", or that agents have enough data to come up with reliable estimate of some probability law of the evolution of some parameter (e.g. dynamic macroeconomics). Game theory has adopted the Harsanyi doctrine that difference in beliefs should be chased down to differences in information, or, in other words that two agents with the same information should hold the same beliefs.

In this paper, we take the view that, in many decision problems, the relevant information the agent has should be explicitly modelled, while allowing for more general form of information than the precise probabilistic information assumed in von Neumann and Morgenstern (1947). A gain from such an approach is that it makes it easy to compare the choice behavior of an agent in a given information state with the same agent's choice in a different information state. As an example, consider a doctor having to do a diagnosis on a patient. The diagnosis and the ensuing treatment could well be different according to whether the doctor has a long medical record about the patient or has no information whatsoever about him. Thus, while the available actions (say different treatments) are the same in these two situations, they still differ by the amount of information the decision maker has. Another feature of this example is that, loosely speaking, the doctor will most probably prefer to deal with the "known" patient rather than with the "unknown" one, thereby showing a preference for acting in contexts in which information is of higher quality. We thus model a decision problem as the choice of an act given an "informational state". We hence postulate that the decision maker has preferences over pairs (act, information), with the idea that when comparing a given act in two distinct informational states, he'll prefer the situation in which the information is more precise.

We thus are left to define more precisely what we call an informational state. First, we assume as usual that the sources of uncertainty can be captured by a set of states of nature. Second, we concentrate our attention on situations in which the information can be represented by a set of probability distributions over the state space, together with an anchor or reference point. To illustrate this idea, take Ellsberg three-color urn experiment (Ellsberg (1961)), in which the decision maker is told there are 30 Red balls and 60 Blue or Green balls in the urn, according to the draw of a roulette wheel (with number from 0 to 60) that gives the number of Blue balls. Then, the set of priors that is appropriate to model the available information is simply the set of all probability distributions that place 1/3 on Red. The anchor of this set is also rather naturally the distribution $(1/3, 1/3, 1/3)$. Our main modelling assumption, that the decision maker has preferences over pairs (act, information) would be in this example to assert that the decision maker is able to compare say "betting on Blue when the information is as above" versus "betting on Blue when the composition of the urn is known to be (1/3,1/6,1/2)". This notion of imprecise information represented by a set of priors and an anchor is the same as the notion discussed in Hansen et alii (2001) and Wang (2001). Hansen and Sargent (2002) thus provide another instance in which this modelling approach is natural, reflecting "agents' fear of model misspecification":

> (...) the agent regards the model [for macroeconomists, a discrete time model is a probability distribution over a sequence of vectors] as an approximation to another unknown "true" model. The agent thinks his model is a good approximation in the sense that the data are generated by another model that belongs to a vaguely specified *set* of models near the approximating model. Hansen and Sargent (2002), Introduction to *Elements of Robust Control and Filtering for Macroeconomics*.

With this setting in place, we suggest a partial order on the sets of priors with a common anchor that reflects the degree of uncertainty, or imprecision in the information, the decision maker faces. The definition of this order, independent from the agent's preferences paves the way towards a theory of uncertainty aversion in which uncertainty has an objective component. Although the setting is restricted to situations that can be represented through a set of priors and an anchor, we believe that the approach we develop is a first step towards a definition of uncertainty aversion based on objective characteristics, in the same way that risk aversion is based on second order stochastic dominance. The order we suggest is now described. Intuitively, assuming that the information available to the decision maker can be expressed by a set of priors, the "size" of this set seems to be a good candidate to measure the imprecision of the information: if the information available enables the decision maker to reduce the set of compatible probabilities, then one can say that imprecision has objectively decreased. To be able to translate this reduction of uncertainty in choice behavior, one needs to restrict the comparison between families of probabilities that have a common anchor. As in situations of risk in which the definition of risk aversion is based on the comparison of two differently spread distributions that have a common mean, our definition of aversion towards imprecision relies on the decision maker comparing two sets of probabilities that have the same anchor.

In our main theorem we establish that a decision maker exhibiting aversion towards imprecision acts as if evaluating an act by the minimum over a *revealed* set of probability distributions of its expected utility. The revealed set of distributions is a subset of the set of objectively admissible probability distributions. More precisely, we reveal a coefficient of pessimism which is the degree to which the decision maker keeps all the objective distributions. An extremely pessimistic agent will keep the entire set, that is the revealed set of priors will be equal to the initial set of admissible priors. Conversely, a decision maker whose choices are not be affected by the imprecision of the situation reduces any prior set of probability distributions to the maximum and ends up acting only on the basis of the reference distribution. We then further characterize the revealed set of priors. Under somewhat stronger axiom, it is shown that is consists of a uniform shrinkage or contraction of the initial set around its anchor. The coefficient of this contraction can then be taken as an index of pessimism.

Why might our result be of interest? Our main contribution in the paper is to provide a link between the information available to the decision maker and his revealed set of beliefs. As we mentioned, the decision criterion we axiomatize is less extreme than taking the minimum expected utility over the entire set of priors compatible with the information. This answers a criticism often formulated against the multiple prior maximin type of approach that it is strongly biased towards extreme conservatism of the decisions selected. Take for instance the global warming problem. Scientific evidence has somewhat restricted the set of possible values for important parameters, without being able, at this stage, to actually assess what are the exact effect of emission of various gas on the average temperature. Taking this evidence into account and applying the maxmin expected utility approach would then "uniquely" determine the optimal (conservative) environmental policy, leaving no room for any influence of the society attitude towards uncertainty. In our setting however, the attitude towards imprecision of the scientific evidence is an important element that, together with the evidence itself, dictates the choice of the optimal environmental policy. Different societies, with different degree of imprecision aversion, will choose different policies. More generally, we believe that modelling scientific uncertainty requires the type of approach formulated here, as it seems difficult to deal with such uncertainties in a probabilistic fashion: what does it mean for a scientific theory to be valid with probability 1/3? In approach on the other hand, one can associate a prior with a theory (the reference prior being the dominant theory at the time) and simply consider all theories (that have passed some minimal adequacy tests) as possible.

Another important consequence of the approach followed in the paper is to be able to perform comparative statics exercise, in which the imprecision of the information has changed. This is indeed an important advantage over a purely subjectivist approach. Much of the results in the literature on portfolio choice (under risk) for instance are based on some comparative statics on the riskiness of the situation or on the degree of risk aversion. An application of the results of this paper would be precisely to revisit some of this literature assuming imprecision. Interestingly, the setting developed might provide some insights into, for example, the revision of the optimal composition of a portfolio when market uncertainty increases or decreases.

Relationship with the literature. The Ellsberg experiments (Ellsberg (1961)) have established that decision makers behave differently when facing a problem with uncertainty than when facing a problem under risk, i.e., in which a sharp probabilistic information is given. This observation has led to a whole strand of literature that axiomatizes models of decision under uncertainty that can represent the decision maker's aversion to uncertainty (Schmeidler (1989), Gilboa and Schmeidler (1989), Jaffray (1989)). Most of this literature follows the lead of Savage (1954), in that the axiomatizations proposed are cast entirely in behavioral terms: the revealed non-probabilistic beliefs are entirely subjective and are not explicitly related to any prior "objective" information . More recently, Epstein and Zhang (2001) go one step further and define in purely behavioral terms what it means for an event to be considered "uncertain" by a decision maker. It is interesting to note that the motivation on which all these constructions rest –the Ellsberg experiments– is one in which the objective information given to the decision makers should clearly identify the set of beliefs compatible with that information.

A few exceptions to this subjectivist approach of uncertainty have to be mentioned. Jaffray (1989) assumed that the information the decision maker has can be represented by belief functions and that the decision maker has preferences over such object (rather than mere lotteries as in von Neumann and Morgenstern (1947)). Imposing an independence axiom on these objects he obtained a representation based on the linear combination of a minimum and a maximum of expected utilities. Another approach is based on the use of two-stage lotteries. Segal (1987) shows that relaxing the reduction of compound lotteries axiom provide an explanation for the Allais and Ellsberg paradoxes. In Segal (1990), rank dependent expected utility is axiomatized *via* choice behavior over two-stage lotteries.

As mentioned above, Handsen and Sargent (2002) develop a model of robust control theory where the reference model (a probability distribution) is given but not precisely known to the agent, who therefore considers possible a set of priors around this reference point. Specifically, agents have an estimation of the "true" model and, as econometrician would do, consider a range of values around that estimate as possible, to take into account the possibility that the true model is in fact different from the estimate. They hence propose a way to deal with objective information that might not be as crisp as a probability distribution but, rather, a set of probability distributions together with a baseline scenario that is given by a probability distribution. Hence, our setup is very similar to theirs. They however do not provide an axiomatic foundation of the decision criterion retained. This axiomatization is done in Wang (2001) who studied the link between this theory and Gilboa and Schmeidler (1989) representation. In doing so, he axiomatizes the same functional form as in Gilboa and Schmeidler taking their set of priors as objectively given: the objective of the agent is then to maximize the minimum of its expected utility computed with all the probability distributions in the set of (objective) priors. This model then has the feature that agents are extremely averse to the imprecision in their probabilistic information. Indeed, taking the worst possible scenario to evaluate each act might be a bit too extreme in many circumstances. The axiomatization of Hansen et alii (2001) criterion (based on a principle of minimization of entropy) in Wang (2001) is also based on the same idea of extreme uncertainty aversion. Hence, compared to Wang (2001), our approach uses a setting that he called "multi-prior with a reference point" and develops within this setting an axiomatization à la Gilboa and Schmeidler (1989) that is weaker in the sense explained above.

Organization of the paper. The paper is organized as follows. In section 2, we introduce our informational setup, and define a partial order on the imprecision of information, independent of the decision maker's preferences. In section 3, we introduce a usual choice theoretic framework that we link with our informational setup. A first step is then to simply recast Gilboa and Schmeidler (1989) theorem in our setup, to obtain a first representation theorem, in which the revealed set of priors is explicitly linked to the prior information. In section 4, we take advantage of our informational setting and propose an axiom called "aversion towards imprecision" that simply states that agents are averse to increases in the imprecision (defined in section 2). This allows us (together with a few other axioms) to replace the "uncertainty aversion" of Gilboa and Schmeidler, to come up with our main representation result. Next, we provide a further characterization of the notion of pessimism in our setting. The last section contains some concluding remarks. Proofs are gathered in an appendix.

2 Sets of priors with an anchor

In this section, we set up our general framework for representing uncertainty. As mentioned in the introduction, we assume that uncertainty is represented through a family of probability distributions together with a reference prior (see Wang (2001)). We'll use interchangeably the term anchor and center for this reference prior. This anchor is a prior that belongs to (convex hull of) the given set of priors and has a particular salience in the decision problem at hand. The decision problems the decision maker face can hence be decomposed in two components: first the possible action he might take and, second, the available information about the sources of uncertainty. In this section we concentrate on this second aspect. Although unusual in the literature, this approach is rather intuitive to represent, for instance, Ellsberg type of experiments as well as changes in the quality of the available information.

2.1 A general setting and some examples

We start with a general description of uncertainty. Throughout the paper the state space S will be assumed to be equal to $\mathbb{N}$. Let $\Sigma = 2^{\mathbb{N}}$ and let $\Delta(S)$ be the simplex on S. We represent uncertainty *via* a closed set of possible probability distributions over S, that is, as a closed subset $\mathcal{P}$ of $\Delta(S)$. This set consists of all the probability distributions that are compatible with the available information. We consider finite settings in the sense that the number of relevant states of nature for a given problem is finite: the set $S(\mathcal{P}) = \cup_{p \in \mathcal{P}} Supp(p)$ is finite. Let $\mathcal{C}$ be the set of set of priors having this property (i.e., closed sets of priors with finite support).

On top of this set of priors, the information available about the situation at hand allows one to identify a reference prior, i.e., a probability distribution over S that is (explicitly or by default) the baseline scenario. A *situation* will be the given of a pair $[\mathcal{P}, c]$ of a set of priors in

$\mathcal{C}$, together with an anchor, such that $c \in \overline{co}(\mathcal{P})$, where $\overline{co}(A)$ is the closed convex hull of A for any set A. Let $\mathcal{S}$ be the set of all such possible situations.

To start with well-known examples, consider Ellsberg's two experiments, which are the usual motivation for studying models of decision under uncertainty that cannot be reduced to decision under risk.

Example 1 *(Ellsberg's two color urns) Consider a decision maker facing two urns containing a hundred balls (either black or white). He's told that there are 50 white balls and 50 black balls in the first urn, while the proportion of each color in the second urn is unknown. In both cases there are two states of nature: "the ball drawn is white" and "the ball drawn is black". The information about the known urn can be represented by the single probability measure* $(.5, .5)$. *A natural description of the second urn is to consider that all probability distributions over black and white are possible, i.e., the set of priors is the entire simplex. The distribution* $(.5, .5)$ *is also a natural anchor for this problem*

Observe in particular that the typical Bayesian case (e.g., the urn with a known proportion of white and black balls in example 1) is a particular case, in which the set of priors reduces to a single probability distribution.

Example 2 *(Ellsberg's three color urn) Consider a decision maker who has to bet over the color of a ball drawn from an urn that contains 30 red balls and 60 green and blue balls in unspecified proportions. The decision maker hence knows that* $\Pr(\mathit{Red}) = \frac{1}{3}$, *and nothing else, which constitutes the set of priors compatible with the available information. The center of this set is the distribution* $(1/3, 1/3, 1/3)$.

In the two examples above, it seems natural to appeal to symmetry considerations to determine the anchor of the set of priors. However, this is not necessarily the case. The following example captures the idea, developed by Handsen and Sargent (2002) that agents might only have estimates of some important parameters (the law of motion of the system in a macroeconomic setting).

Example 3 *(Statistical inference) Consider the common practice of sampling a given population to assess the probability* p *of appearance of a particular feature in this population. The common practice in statistics is to consider as "possible" all the parameters value that fall into the 95% confidence interval around the estimate of* p. *Thus, the set of priors is this interval and the anchor the estimate of* p.

Another example that can be treated in our example is the case of two-stage lotteries. Segal (1987) in particular argued that both Allais and Ellsberg paradoxes could be explained through a relaxation of the reduction of compound lotteries axiom. His setup can be seen as a particular case of ours, since he works with second order probabilities. Thus, our set of priors can be taken to be the support of this second order distribution and the anchor is simply the mean of the distribution.

Finally, consider the case in which the decision maker asks their opinion to different experts, who come up with different assessments of the situation.

Example 4 *(Aggregation of expert's opinions) Assume the decision maker asks two equally reliable experts to assess the probability of occurrence of a given event. The first expert comes up with the evaluation* $(1, 0)$ *(the event will occur with probability 1) while the second expert comes up with the evaluation* $(0, 1)$ *(the event will occur with probability 0). The decision maker, acknowledging the disagreement of these two experts will keep these two distributions as possible. The center of this set might depend on the reliability of the experts. If both are equally reliable, the center is simply the distribution (1/2,1/2).*

This example can be extended to models representing situations in which scientific theories compete for explaining a particular phenomenon. Scientific theories are then viewed as probability distributions over a state space. The set of priors therefore amounts to the set of theories and the anchor is then the dominant theory, challenged by the new ones.

2.2 Comparison of imprecise information

The representation of uncertainty through sets of priors has a direct implication for the comparison of two situations. Very naturally, a situation will be considered more imprecise than an other if the set of probabilities considered possible in the second situation is included in the set of probabilities in the first situation.

Definition 1 *Let* $\mathcal{P}_1, \mathcal{P}_2 \in \mathcal{C}$*. The set* $\mathcal{P}_1 \subseteq \Delta(S)$ *is more imprecise than the set* $\mathcal{P}_2$ *if* $\overline{co}\,(\mathcal{P}_1) \supseteq \overline{co}\,(\mathcal{P}_2)$.

This definition calls for several comments. First, the "amount" of imprecision in a given situation depends on the size of the family of probabilities compatible with the information. However, mere set inclusion is not enough in our view. Indeed, our motivation here is to characterize an objective notion of imprecision, with the idea that a decision maker will always prefer a given decision in the least imprecise situation. Hence, our definition of what it means for a situation to be more precise than another has to be guided by what one would intuitively consider as the choice behavior of a decision maker in more or less imprecise situations. In other words, we need to define the notion of "more imprecise than" keeping in mind the type of choice behavior we want to analyze. Thus, although it seems sensible to say that the situation in which the decision maker knows that there are 1 white and 99 black balls in an urn is more precise than the situation in which the decision maker has no information whatsoever on the proportion of white and black balls, it seems also sensible to assume that the decision maker would prefer to bet on white in the unknown urn rather than in the known urn. This gives rise to a more specific definition of "more imprecise than":

Definition 2 *Let $\mathcal{P}_1, \mathcal{P}_2 \in \mathcal{C}$ and $c_1, c_2 \in \Delta(S)$. The situation $[\mathcal{P}_1, c_1]$ is a center preserving increase in imprecision of the situation $[\mathcal{P}_2, c_2]$ if*

1. *$\mathcal{P}_1$ is more imprecise than $\mathcal{P}_2$, i.e., $\overline{co}(\mathcal{P}_1) \supseteq \overline{co}(\mathcal{P}_2)$.*
2. *$c_1 = c_2$.*

This definition captures the intuition that an urn with 100 balls, in which it is known that there are at least 20 white and 20 black balls is more precise than an urn whose composition is unknown (but for the fact that it is made of black and white balls). Another intuitive feature that is embedded in our definition is slightly more subtle. Consider Ellsberg's experiments again. In the two color urns experiment, consider an experiment in which there are two balls in each urn. The situation can be described by the set $\mathcal{P}_1 = \{(1,0),(0,1)\}$ of probability distributions over $\{white, black\}$ and center $(1/2, 1/2)$. Now consider the same story, but with three balls in each urn. The set of compatible priors is $\mathcal{P}_2 = \{(1,0),(2/3,1/3),(1/3,2/3),(0,1)\}$ and its center is $(1/2, 1/2)$. Intuitively, the imprecision of the two situations are the same, as our intuition tells us that the number of balls is immaterial here, and indeed our definition asserts precisely that these two situations are identical as far as imprecision is concerned, since the convex hull of the two sets coincide and their centers are identical.

We define now the notion of a *contraction* of a situation, which will be used in section 3 to exhibit the revealed beliefs of the decision maker from the objective data. The contraction of a situation corresponds to a center preserving decrease in imprecision.

Definition 3 *For all $[\mathcal{P}, c] \in \mathcal{S}$ and $\alpha \in]0,1]$, call the α-contraction of $[\mathcal{P}, c]$ the situation $[\mathcal{P}^\alpha, c]$ with $\mathcal{P}^\alpha = \{p^\alpha \in \Delta(S) | p^\alpha = \alpha p + (1-\alpha)c, p \in \mathcal{P}\}$.*

To illustrate the notions of uncertainty and contraction as defined above, consider Ellsberg's three color urn again.

Example 5 *(example 2 continued). We compare here the original situation (no information on the proportion of blue and green balls) to the situation in which the information is that there are at least 20 green balls and 20 blue balls, the remaining 20 being either green or blue. More precisely, let p_R, p_G, and p_B denote respectively the probability of drawing a red, green, or blue ball. Then, $\mathcal{P}_1 = \{p \in \Delta(\{R,G,B\}) | p_R = 1/3\}$ and $c_1 = (1/3, 1/3, 1/3)$ while $\mathcal{P}_2 = \{p \in \Delta(\{R,G,B\}) | p_R = 1/3, p(G) \geq 2/9, p(B) \geq 2/9\}$ and $c_2 = (1/3, 1/3, 1/3)$.*

Clearly, $[\mathcal{P}_1, c_1]$ is a center preserving decrease in imprecision of $[\mathcal{P}_2, c_2]$. It is also easy to show that $[\mathcal{P}_2, c_2]$ is a 1/3-contraction of $[\mathcal{P}_1, c_1]$.

Graphically, this can be represented in the simplex as follows:

The particular form of contraction we defined amounts to a reduction in imprecision.

Lemma 1 *Let* $[\mathcal{P}, c] \in \mathcal{S}$. *Then,* $[\mathcal{P}^{\alpha}, c]$ *is a center preserving decrease in imprecision of* $[\mathcal{P}, c]$.

With the description of the informational environment in place we can now turn to the modelling of choice behavior and preference representation.

3 Maxmin expected utility revisited

We now turn to the description and representation of a decision maker's preferences in a situation of uncertainty. In this section, we develop a decision making setup in which we model explicitly the decision maker's imprecise information and the use he makes of it. We first introduce our general decision theoretic setting and then recast Gilboa and Schmeidler (1989)'s analysis in our terms, thus relating the revealed set of priors to the available information.

3.1 Setup

As Schmeidler (1989), Gilboa and Schmeidler (1989), among others, we use the framework of Anscombe and Aumann (1963). This is mostly for the sake of simplicity, as it enables us to build on their representation theorem. Let X be a set (the set of outcomes) and let Y be the set of distributions over X with finite supports (roulette lotteries).We recall that $S = \mathbb{N}$ is the state space, and $\Sigma = 2^{\mathbb{N}}$. An act f is a mapping from S to Y. We denote by $\mathcal{A}$ the set of acts (horse lotteries). Finally, let k_y be the constant act that gives the distribution $y \in Y$ in all states, and $\mathcal{A}^c$ the set of constant acts. We denote δ_x the lottery giving $x \in X$ with probability one.

We now describe the decision maker's preferences in the setup we introduced, thus taking into account explicitly the probabilistic information (that could be imprecise) the decision maker has when he takes a decision. The decision maker's preferences is a binary relation $\succeq$ over $\mathcal{A} \times \mathcal{S}$, that is on couples $(f, [\mathcal{P}, c])$. As usual, $\succ$ and $\sim$ denote the asymmetric and symmetric parts, respectively, of $\succeq$. In Ellsberg's two urns example, the preference of the decision maker to bet on, say, Black in the known urn rather than on Black in the urn with an unknown proportion of black and white balls would be written: $(f, [\mathcal{P}_1, c_1]) \succeq (f, [\mathcal{P}_2, c_2])$ where f is the act "betting on Black", $\mathcal{P}_1 = \{(1/2, 1/2)\} = c_1$ and $\mathcal{P}_1 = \Delta(\{black, white\})$ and $c_2 = c_1$.

3.2 An extension of Gilboa and Schmeidler's multiple prior representation

We start with an extension of the multiple prior model of Gilboa and Schmeidler (1989) taking into account the information given to the decision maker. The following six axioms are mere restatements of axioms proposed by Gilboa and Schmeidler (1989) in our setup.

Axiom 1 *(Weak order)* $\succeq$ *is complete and transitive.*

Axiom 2 *(Certainty-Independence) For all* $f, g \in \mathcal{A}$, $h \in \mathcal{A}^c$, $[\mathcal{P}, c] \in \mathcal{S}$, $\alpha \in]0,1[$, $(f, [\mathcal{P}, c]) \succ (g, [\mathcal{P}, c]) \Leftrightarrow (\alpha f + (1-\alpha)h, [\mathcal{P}, c]) \succ (\alpha g + (1-\alpha)h, [\mathcal{P}, c])$.

Axiom 3 *(Continuity) For all* f, g, $h \in \mathcal{A}$, *and all* $[\mathcal{P}, c] \in \mathcal{S}$, *if* $(f, [\mathcal{P}, c]) \succ (g, [\mathcal{P}, c]) \succ (h, [\mathcal{P}, c])$, *then there exist* α *and* β *in* $]0,1[$ *such that* $(\alpha f + (1-\alpha)h, [\mathcal{P}, c]) \succ (g, [\mathcal{P}, c]) \succ (\beta f + (1-\alpha)h, [\mathcal{P}, c])$.

Observe that $\succeq$ induces a preference relation $\succeq^{\ell}_{[\mathcal{P},c]}$ on Y which is simply the restriction of $\succeq$ on $\mathcal{A}^c \times \{[\mathcal{P}, c]\}$.

Axiom 4 *(Monotonicity) For all* $f, g \in \mathcal{A}$, *and all* $[\mathcal{P}, c] \in \mathcal{S}$, *if* $f(s) \succeq^{\ell}_{[\mathcal{P},c]} g(s)$ *for all* $s \in S(\mathcal{P})$, *then* $(f, [\mathcal{P}, c]) \succeq (g, [\mathcal{P}, c])$.

Axiom 5 *(Non-degeneracy) For all* $[\mathcal{P}, c] \in \mathcal{S}$, *there exist* f,$g \in \mathcal{A}$ *such that* $(f, [\mathcal{P}, c]) \succ (g, [\mathcal{P}, c])$.

Axiom 6 *(Uncertainty aversion) For all* $f, g \in \mathcal{A}$, $[\mathcal{P}, c] \in \mathcal{S}$, *and all* $\alpha \in]0,1[$, *if* $(f, [\mathcal{P}, c]) \sim (g, [\mathcal{P}, c])$ *then* $(\alpha f + (1-\alpha)g, [\mathcal{P}, c]) \succeq (f, [\mathcal{P}, c])$.

Gilboa and Schmeidler have proved that Axioms 1 to 6 hold if, and only if, for all $[\mathcal{P}, c]$ there exist an unique (up to a positive linear transformation) affine function $u_{[\mathcal{P},c]} : Y \to \mathbb{R}$, and an unique, non-empty, closed and convex set $\mathcal{F}_{[\mathcal{P},c]}$ of probability measures on 2^S, such that for all $f, g \in \mathcal{A}$, $(f, [\mathcal{P}, c]) \succeq (g, [\mathcal{P}, c])$ if and only if:

$$\min_{p \in \mathcal{F}_{[\mathcal{P},c]}} \int u_{[\mathcal{P},c]} \circ f dp \geq \min_{p \in \mathcal{F}_{[\mathcal{P},c]}} \int u_{[\mathcal{P},c]} \circ g dp.$$

However, this result falls short of a representation theorem in our setup since it considers only a given $[\mathcal{P}, c]$. Our setup hence calls for an extension of Gilboa and Schmeidler's axioms, to be able to compare acts in possibly different states of information, i.e., in situations characterized by different degrees of imprecision.

The first new axiom we introduce requires some further definitions. The aim of this construction is to ensure that the only relevant information upon which the decision maker is acting is the set of induced probability over outcomes and not the states of the world themselves.

For any φ onto mapping from S to S (i.e $\varphi(S) = S$), for any $f \in \mathcal{A}$, we say that f is φ-measurable if $f(s) = f(s')$ for all $s, s' \in S$ such that $\varphi(s) = \varphi(s')$. If f is φ-measurable, define f^{φ} by $f^{\varphi}(s) = f(s')$ where $s' \in \varphi^{-1}(s)$ for all $s \in S$.

For any φ onto mapping from S to S, for any $p \in \Delta(S)$ and $[\mathcal{P}, c] \in \mathcal{S}$, define p^{φ} by $p^{\varphi}(s) = p(\varphi^{-1}(s))$ for all $s \in S$ and $\mathcal{P}^{\varphi}$ by $\mathcal{P}^{\varphi} = \{q \in \Delta(S) | q = p^{\varphi}, p \in \mathcal{P}\}$. If φ is a bijection, note that for all $p \in \Delta(S)$, there is a unique $q \in \Delta(S)$ such that $q^{\varphi} = p$.

Definition 4 *(Equivalence) For all $f, g \in \mathcal{A}$, $[\mathcal{P}_1, c_1], [\mathcal{P}_2, c_2] \in \mathcal{S}$, $(f, [\mathcal{P}_1, c_1])$ and $(g, [\mathcal{P}_2, c_2])$ are equivalent if there exist φ_1 and φ_2 two onto mappings from S to S, such that f is φ_1-measurable, g is φ_2-measurable, $\mathcal{P}_1^{\varphi_1} = \mathcal{P}_2^{\varphi_2}$, $c^{\varphi_1} = c^{\varphi_2}$, and $f^{\varphi_1} = g^{\varphi_2}$.*

In particular, remark that if f is φ-measurable, then $(f, [\mathcal{P}, c])$ is equivalent to $(f^{\varphi}, [\mathcal{P}^{\varphi}, c^{\varphi}])$.

Example 6 *Consider the acts f and g defined by $f(1) = 1$ and $f(s) = 0$ for $s \geq 2$ and $g(1) = g(2) = 1$, and $g(s) = 0$ for $s \geq 3$. Consider $\mathcal{P}_1 = \{p \in \Delta(S) | p(1) = .5,\ p(2) + p(3) = .5\}$ and $c_1 = (1/2, 1/4, 1/4)$ on the one hand and $\mathcal{P}_2 = \{p \in \Delta(S) | p(1) + p(2) = .5,\ p(3) = .5\}$ and $c_2 = (1/4, 1/4, 1/2)$ on the other hand. It is easy to see that $(f, [\mathcal{P}_1, c_1])$ and $(g, [\mathcal{P}_2, c_2])$ are equivalent. Indeed, define $\varphi_1 : S \to S$ by $\varphi_1(1) = 1$, $\varphi_1(2) = \varphi_1(3) = 2$ and $\varphi_1(s) = s - 1$ for $s \geq 4$ and $\varphi_2 : S \to S$ by $\varphi_2(1) = \varphi_1(2) = 1$ and $\varphi_2(s) = s - 1$ for $s \geq 3$. Then, it is straightforward to see that f is φ_1-measurable, g is φ_2-measurable and that $\mathcal{P}_1^{\varphi_1} = \mathcal{P}_2^{\varphi_2}$, $c_1^{\varphi_1} = c_2^{\varphi_2}$, and $f^{\varphi_1} = g^{\varphi_2}$.*

The next axiom states that the decision maker is indifferent among two equivalent representations of a couple (act, situation).

Axiom 7 *(Equivalence indifference) For all $f, g \in \mathcal{A}$, $[\mathcal{P}_1, c_1], [\mathcal{P}_2, c_2] \in \mathcal{S}$, if $(f, [\mathcal{P}_1, c_1])$ and $(g, [\mathcal{P}_2, c_2])$ are equivalent then $(f, [\mathcal{P}_1, c_1]) \sim (g, [\mathcal{P}_2, c_2])$.*

This axiom implies in particular that for constant acts, the statistical information given does not matter, as shown in the following lemma.

Lemma 2 *Axiom 7 implies that for all $f \in \mathcal{A}^c$, for all $[\mathcal{P}_1, c_1], [\mathcal{P}_2, c_2] \in \mathcal{C}$, $(f, [\mathcal{P}_1, c_1]) \sim (f, [\mathcal{P}_2, c_2])$.*

Let $f_{S(\mathcal{P})}g$ denote the act that gives f on $S(\mathcal{P})$ and g on $S \backslash S(\mathcal{P})$. The following axiom states that what happens outside of the "support of the set of priors" (i.e., outside of $S(\mathcal{P})$) is irrelevant for the decision maker.

Axiom 8 *(Irrelevant states) For all $f, g \in \mathcal{A}$, $[\mathcal{P}, c] \in \mathcal{S}$, $(f, [\mathcal{P}, c]) \sim (f_{S(\mathcal{P})}g, [\mathcal{P}, c])$*

We can then state a first extension of Gilboa and Schmeidler's (1989) result.

Theorem 1 *Axioms 1 to 8 hold iff there exists an unique (up to a positive linear transformation) affine function $u : Y \to \mathbb{R}$, and for all $[\mathcal{P}_i, c_i] \in \mathcal{S}$, there exists a unique, non-empty, closed and convex set $\mathcal{F}_{[\mathcal{P}_i,c_i]}$ of finitely additive probability measures on 2^S such that:*

1. *For all $p \in \mathcal{F}_{[\mathcal{P},c]}$, $p(S(\mathcal{P})) = 1$,*
2. *For all φ onto mapping from S to S, $\mathcal{F}_{[\mathcal{P}^\varphi,c^\varphi]} = \{p^\varphi | p \in \mathcal{F}_{[\mathcal{P},c]}\}$,*

and such that for all $f, g \in \mathcal{A}$, $(f, [\mathcal{P}_i, c_i]) \succeq (g, [\mathcal{P}_j, c_j])$ iff:

$$\min_{p \in \mathcal{F}_{[\mathcal{P}_i,c_i]}} \int u \circ f dp \geq \min_{p \in \mathcal{F}_{[\mathcal{P}_j,c_j]}} \int u \circ g dp.$$

In this representation theorem, $\mathcal{F}_i$ is the revealed set of probabilities associated to the information $[\mathcal{P}_i, c_i]$. Note that the revealed family does not come with an anchor. To illustrate this theorem, go back to Ellsberg two-urn experiment. In the known urn, our representation result forces the revealed beliefs to be equal to the probability distribution $(.5, .5)$. In the unknown urn, the revealed family can be any family of probability distributions over the state space $\{black, white\}$. For instance, it could be equal to $\{(p, 1-p) | 3/4 \geq p \geq 1/4\}$, or to $\{(p, 1-p) | 2/3 \geq p \geq 1/2\}$.

This theorem provides a representation of preferences that has the advantage of being more flexible than Gilboa and Schmeidler (1989) result applied to objective sets of priors as in Wang (2001). Indeed, the decision maker behaves as a maxmin multiple prior, but not necessarily with respect to the entire family of probability distributions.

Finally, the theorem rests essentially on axioms that are similar to the ones in Gilboa and Schmeidler (1989). In particular, we used their uncertainty aversion axiom. However, the machinery we put in place in the previous section was precisely designed to be able to replace this axiom, based on the mixture of acts, by the comparison of two situations that can be ranked according to our (partial) order of "more imprecise than".

4 A representation theorem for imprecision averse decision makers

This section contains the main result of the paper. We build on the previous theorem in which we now replace Gilboa and Schmeidler's axiom of uncertainty aversion by an axiom that describes the decision maker aversion towards increases in the imprecision of the information at hand. In order to prove our theorem we however need to add a few other axioms, that require defining some operations on sets of probabilities, to which we now turn.

4.1 Replication and set mixture

We define here two notions, namely that of replication and of set mixture that will prove useful in the sequel.

Definition 5 *(Replication) Let $[\mathcal{P}, c] \in \mathcal{S}$, $\alpha \in [0,1]$, and $S' \subset \mathbb{N}$ such that $S' \cap S(\mathcal{P}) = \emptyset$. Let φ be a bijection from S to S such that $\varphi(S(\mathcal{P})) = S'$. The (α, φ)-*replication *of $[\mathcal{P}, c]$ denoted $[\mathcal{P}_R^{(\alpha,\varphi)}, c_R^{(\alpha,\varphi)}]$ is given by:*

$$\mathcal{P}_R^{(\alpha,\varphi)} = \{q \in \Delta(S(\mathcal{P}) \cup S') | q = \alpha p + (1-\alpha)p^{\varphi}, p \in \mathcal{P}\}$$

and

$$c_R^{(\alpha,\varphi)} = \alpha c + (1-\alpha)c^{\varphi}$$

In the definition, $q = \alpha p + (1-\alpha)p^{\varphi}$ is the probability distribution on $S(\mathcal{P}) \cup S'$ defined by $q(s) = \alpha p(s)$ if $s \in S(\mathcal{P})$ and $q(s) = (1-\alpha)p^{\varphi}(s)$ if $s \in S'$.
Note that $\mathcal{P}_R^{(1,\varphi)} = \mathcal{P}$ and $\mathcal{P}_R^{(0,\varphi)} = \mathcal{P}^{\varphi}$.

Example 7 *Consider an urn with one ball, that could be black or white. Consider the "replication" of this urn, with one ball that could be red or green. Replication could be thought of as, for instance, taking the initial urn, painting the ball in red if it was originally black or in green if it was originally white. Define $\varphi(B) = R$ and $\varphi(W) = G$. Then the $1/2$-replication is the probability density that puts weight $1/2$ on the two events (B, R) and (W, G) and zero elsewhere. It corresponds to the description of the urn that is composed of putting together the two urns described.*

The next operation on sets of probability distributions we introduce is the mere mixture of probabilities in the set.

Definition 6 *(Set mixture) Let $[\mathcal{P}_i, c_i], [\mathcal{P}_j, c_j] \in \mathcal{S}$, and $\alpha \in [0,1]$, the α-*mixture *of $([\mathcal{P}_i, c_i], [\mathcal{P}_j, c_j])$, denoted $\left[\mathcal{P}_M^{(\alpha,i,j)}, c_M^{(\alpha,i,j)}\right]$, is given by:*

$$\mathcal{P}_M^{(\alpha,i,j)} = \{q \in \Delta(S(\mathcal{P}_i) \cup S(\mathcal{P}_j)) | q = \alpha p_i + (1-\alpha)p_j, p_i \in \mathcal{P}_i, p_j \in \mathcal{P}_j\}$$

and

$$c_M^{(\alpha,i,j)} = \alpha c_i + (1-\alpha)c_j$$

We will also sometimes write $\left[\mathcal{P}_M^{(\alpha,i,j)}, c_M^{(\alpha,i,j)}\right] = \alpha[\mathcal{P}_i, c_i] + [\mathcal{P}_j, c_j]$.

Remark 1 *A particular instance of set mixture, that will be heavily used in the sequel, is the mixture of $[\mathcal{P}, c]$ with $[\mathcal{P}^{\varphi}, c^{\varphi}]$. We will denote it $\left[\mathcal{P}_M^{(\alpha,\varphi)}, c_M^{(\alpha,\varphi)}\right]$.*

It is obvious to check that the center of the replicated family is the same as the center of this mixed family, i.e., $c_R^{(\alpha,\varphi)} = c_M^{(\alpha,\varphi)}$.

Example 8 *Consider an urn with a hundred balls that could be black or white and another urn with a hundred balls that could be red or green. The $1/4$-mixture is situation obtained by building an urn by taking 25 balls from the first urn and 75 from the second urn.*

The next lemma establishes that the set mixture operation yields a situation that is less precise than the situation obtained *via* replication.

Lemma 3 *Let* $[\mathcal{P}, c] \in \mathcal{S}$ *and* $\alpha \in [0,1]$. *Let* φ *be a bijection from* $S(\mathcal{P})$ *to* S', *with* $S' \cap S(\mathcal{P}) = \emptyset$. *Then,* $\left[\mathcal{P}_M^{(\alpha,\varphi)}, c_M^{(\alpha,\varphi)}\right]$ *is a center preserving increase in imprecision of* $[\mathcal{P}_R^{(\alpha,\varphi)}, c_R^{(\alpha,\varphi)}]$

This lemma, establishing that the replication operation decreases imprecision compared to the mixture operation, is at the heart of our next representation result, based on a form of the uncertainty aversion axiom of Gilboa and Schmeidler (1989) that is entirely based on the comparison of the same act in two different informational environments.

4.2 Representation of imprecision averse preferences

Our next step is to replace Gilboa and Schmeidler's axiom of uncertainty aversion. Indeed, there is a natural objective notion of uncertainty aversion in our setup: if a situation becomes more uncertain (in the sense of "center preserving increase in imprecision" as defined above), an uncertainty averse decision maker will prefer any given decision in the less uncertain situation to the same decision in the more uncertain situation. As it turns out, this reformulation of uncertainty aversion, that we name "imprecision aversion" or "aversion towards imprecision", encompasses Gilboa and Schmeidler's notion of uncertainty aversion provided some other natural axioms.

Axiom 9 *(Aversion towards imprecision) For all* $f \in \mathcal{A}$, $[\mathcal{P}_1, c_1], [\mathcal{P}_2, c_2] \in \mathcal{S}$ *such that* $[\mathcal{P}_1, c_1]$ *is a center preserving increase in imprecision of* $[\mathcal{P}_2, c_2]$, $(f, [\mathcal{P}_2, c_2]) \succeq (f, [\mathcal{P}_1, c_1])$.

Remark 2 *Aversion towards imprecision only compares situations with the same anchor (i.e.,* $c_1 = c_2$*).*

An imprecision neutral averse decision maker is such that for all $f \in \mathcal{A}$, $[\mathcal{P}_1, c_1], [\mathcal{P}_2, c_2] \in \mathcal{S}$ such that $[\mathcal{P}_1, c_1]$ is (center preserving) more imprecise than $[\mathcal{P}_2, c_2]$, $(f, [\mathcal{P}_1, c_1]) \sim (f, [\mathcal{P}_2, c_2])$. For instance, in the three-color-urn example, an imprecision averse decision maker would prefer to bet on blue when it is known that the proportion of blue balls is 1/3 than betting on blue when the only information is that the proportion of blue balls is anything between 0 and 2/3, while an imprecision neutral decision maker would be indifferent between these two bets.

Before introducing two further axioms, consider the following example, in which we illustrate how a decision maker "should" react to replication and mixture operations.

Example 9 *(Replication and mixture)*

Consider a decision maker facing two urns containing each a hundred balls. He is told that there is an unknown proportion of black and white balls in urn 1, and an unknown proportion of red and blue balls in urn 2. The decision maker is indifferent between betting on black in urn 1 or on blue in urn 2.

- *The mixture story: urn 3 is built by putting together all the balls of urn 1 and in urn 2. Urn 3 is clearly an $\frac{1}{2}$-mixture of urns 1 and 2. Hence, according to axiom 10 below, the decision maker is indifferent between betting on black and blue in urn 3, betting on black in urn 1 and betting on blue in urn 2, which seems to be a sensible choice.*

- *The replication story: the decision maker is told that in fact, there is exactly the same number of black balls in urn 1 as the number of red balls in urn 2. This new information does not modify his indifference between betting on black in urn 1 or on blue in urn 2. All the balls in urn 1 and 2 are now put together in urn 4. Consider the following bet: a ball is drawn in urn 1 and whatever the color, a non biased coin is tossed; the decision maker wins if the coin lands on tails. The decision maker is indifferent between this last bet and betting on black and blue in urn 4. According to Gilboa and Schmeidler's uncertainty aversion axiom, the decision maker should prefer the coin toss to the initial bets (black in urn 1 on the one hand and blue in urn 2 on the other hand). Urn 4 is clearly an $\frac{1}{2}$-replication of urn 1 (urn 2 being itself a* 1- *replication of urn 1). The last bet correspond to an ex post randomization. The idea is hence that ex post randomization can be equivalently described by replication.*

The two following axioms capture the intuition we had about the decision maker preferences about replication, mixture, and randomization.

Axiom 10 *(Mixture independence axiom) For all $[\mathcal{P}_i, c_i] \in \mathcal{S}$, $i = 1, 2, 3$ such that $(S(\mathcal{P}_1) \cup S(\mathcal{P}_2)) \cap S(\mathcal{P}_3) = \emptyset$, for all $\alpha \in [0, 1)$, and for all $f, g \in \mathcal{A}$ such that $f(s) = g(s)$ for all $s \in S(\mathcal{P}_3)$,*

$$(f, [\mathcal{P}_1, c_1]) \succeq (g, [\mathcal{P}_2, c_2]) \Leftrightarrow \left(f, [\mathcal{P}_M^{(\alpha,1,3)}, c_M^{(\alpha,1,3)}]\right) \succeq \left(g, [\mathcal{P}_R^{(\alpha,2,3)}, c_R^{(\alpha,2,3)}]\right)$$

Recall that in our setup, the decision maker is an expected utility maximizer in situations of risk (i.e., whenever evaluating a lottery). The mixture independence axiom thus expresses the fact that an *ex ante* randomization among different situations does not have any bearing on the decision maker's choice, since such a randomization does not reduce imprecision in any sense.

Axiom 11 *(Replication indifference) For all $[\mathcal{P}, c] \in \mathcal{S}$, for all replication $[\mathcal{P}_R^{(\alpha,\varphi)}, c_R^{(\alpha,\varphi)}]$, for all $f, g \in \mathcal{A}$ such that $f(s) = g(s)$ for all $s \in S \setminus S(\mathcal{P})$,*

$$(\alpha f + (1-\alpha) g, [\mathcal{P}, c]) \sim (f_{S(\mathcal{P})} g^{\varphi}, [\mathcal{P}_R^{(\alpha,\varphi)}, c_R^{(\alpha,\varphi)}])$$

In the previous axiom, observe that $f_{S(\mathcal{P})} g^{\varphi}(s) = f(s) = g(s)$ for all $s \in S \setminus (S(\mathcal{P}) \cup \varphi(S(\mathcal{P})))$.

Lemma 4 *Assume that axioms 7, 8, and 10 hold. Let $[\mathcal{P}, c] \in \mathcal{S}$, $f, g \in \mathcal{A}$, $S' \subset \mathbb{N}$ such that $S' \cap S(\mathcal{P}) = \emptyset$, and φ a bijection from S to S such that $\varphi(S(\mathcal{P})) = S'$. Then,*

$$\forall \alpha \in [0, 1], \quad (f, [\mathcal{P}, c]) \sim (g, [\mathcal{P}, c]) \Rightarrow (f, [\mathcal{P}, c]) \sim (g, [\mathcal{P}, c]) \sim (f_{S(\mathcal{P})} g^{\varphi}, [\mathcal{P}_R^{(\alpha,\varphi)}, c_R^{(\alpha,\varphi)}])$$

In example 9, we had the intuition that replication does reduce imprecision while it is not the case of the mixture operation. The following lemma shows that this intuition is correct. It shows the strong link between Gilboa-Schmeidler's uncertainty aversion axiom and our imprecision aversion axiom: indeed, the lemma shows that the type of behavior that they associate with (subjective) uncertainty aversion can be deduced from a notion of aversion to the imprecision of the information over some important features of the decision problem, that we represent through the fact that a set of probability distributions are compatible with the information.

Lemma 5 *Axioms 7 to 11 imply axioms 2 6.*

Remark that Aversion towards Imprecision (axiom 9) is not necessary to deduce Certainty Independence (axiom 2).

We next introduce an axiom stating a Pareto-like condition. If an act is preferred to another one for any situation in which the information is precise and given by a probability in the set of priors, then the same preference must hold when the information is given by this set.

Axiom 12 *(Pareto) For all $[\mathcal{P}, c] \in \mathcal{S}$, if for all $p \in \Delta(S)$ such that $p \in \mathcal{P}$, we have $(f, [\{p\}, c]) \succeq (g, [\{p\}, c])$, then $(f, [\mathcal{P}, p]) \succeq (g, [\mathcal{P}, p])$.*

Not surprisingly, this axiom implies the axiom "irrelevant state" that asserts that the decision maker does not consider as relevant the part of the act whose payoffs depend on the realization of events which are ruled out by the available information.

Lemma 6 *Axiom 12 implies axiom 8.*

The next theorem is our main result. It provides a characterization of our set of axioms, in which the notion of uncertainty aversion is captured by the aversion towards the information imprecision.

Theorem 2 *Axioms 1, 3 to 5, 7, and 9 to 12 hold if, and only if, there exist an unique (up to a positive linear transformation) affine function $u : Y \to \mathbb{R}$, and for all $[\mathcal{P}_i, c_i] \in \mathcal{C}$, there exist unique, non-empty, closed and convex sets $\mathcal{F}_{[\mathcal{P}_i,c_i]}$ of probability measures on 2^S, satisfying*

1. *$\mathcal{F}_{[\mathcal{P}_i,c_i]} \subseteq \mathcal{P}_i$*
2. *For all φ onto mapping from S to S, $\mathcal{F}_{[\mathcal{P}_i^{\varphi},c_i^{\varphi}]} = \{p^{\varphi} | p \in \mathcal{F}_{[\mathcal{P}_i,c_i]}\}$*
3. *If $[\mathcal{P}_i, c_i], [\mathcal{P}_j, c_j] \in \mathcal{S}$ are such that $S(\mathcal{P}_i) \cap S(\mathcal{P}_j) = \emptyset$, then for all $\alpha \in [0, 1]$,*

$$\mathcal{F}_{\left[\mathcal{P}_M^{(\alpha,i,j)},c_M^{(\alpha,i,j)}\right]} = \alpha\mathcal{F}_{[\mathcal{P}_i,c_i]} + (1-\alpha)\mathcal{F}_{[\mathcal{P}_j,c_j]}$$

4. *If $[\mathcal{P}_i, c_i]$ is a center preserving increase in imprecision of $[\mathcal{P}_j, c_j]$ then $\mathcal{F}_{[\mathcal{P}_i,c_i]} \supseteq \mathcal{F}_{[\mathcal{P}_j,c_j]}$*

5. *For* $[\mathcal{P}, c] \in \mathcal{S}$, *for all replication* $\left[\mathcal{P}_R^{(\alpha,\varphi)}, c_R^{(\alpha,\varphi)}\right]$,

$$\mathcal{F}_{\left[\mathcal{P}_R^{(\alpha,\varphi)}, c_R^{(\alpha,\varphi)}\right]} = \left\{\alpha p + (1-\alpha)p^{\varphi} | p \in \mathcal{F}_{[\mathcal{P},c]}\right\}$$

such that for all $f, g \in \mathcal{A}$, $(f, [\mathcal{P}_i, c_i]) \succeq (g, [\mathcal{P}_j, c_j])$ *if, and only if:*

$$\min_{p \in \mathcal{F}_{[\mathcal{P}_i, c_i]}} \int u \circ f dp \geq \min_{p \in \mathcal{F}_{[\mathcal{P}_j, c_j]}} \int u \circ g dp.$$

The theorem provides an axiomatization of the maxmin multiple prior model where the revealed set of priors is somewhat constrained by the available information. These constraints are reflected in conditions 1 to 5 and put some structure on this revealed set. For instance, given two situations that can be compared according to the partial order "more imprecise than" that we defined in the previous section, it must be the case that the revealed set of priors in the less imprecise situation must be included in the revealed set of priors in the more imprecise situation (condition 4).

4.3 Characterizing pessimism

Our aim in this section is to further specialize the representation theorem obtained above. We do so by strengthening the aversion to imprecision of information axiom.

Axiom 13 *(Dominance) For all* $f, g \in \mathcal{A}$, $[\mathcal{P}_1, c_1], [\mathcal{P}_2, c_2] \in \mathcal{S}$, *if* $(f, [\{c_1\}, c_1]) \succeq (g, [\{c_2\} c_2])$ *and for all* $p \in \mathcal{P}_1$, *there exists* $q \in \mathcal{P}_2$ *such that* $(f, [\{p\}, p]) \succeq (g, [\{q\}, q])$, *then* $(f, [\mathcal{P}_1, c_1]) \succeq (g, [\mathcal{P}_2, c_2])$.

The next lemma shows that this axiom is indeed a strengthening of the aversion to imprecise information axiom.

Lemma 7 *Axiom 13 implies axioms 9 and 12.*

The last representation theorem we derive gives a particular form for the revealed set of priors: under the dominance axiom, it has to be equal to a contraction of the set of compatible priors. The degree of contraction can then be seen as an index of pessimism.

Theorem 3 *Axioms 1, 3 to 5, 7, 10, 11, and 13 hold if, and only if, there exist an unique (up to a positive linear transformation) affine function* $u : Y \to \mathbb{R}$, *and* $\alpha \in [0, 1]$, *such that for all* $[\mathcal{P}_1, c_1], [\mathcal{P}_2, c_2] \in \mathcal{S}$, *for all* $f, g \in \mathcal{A}$,

$$(f, [\mathcal{P}_1, c_1]) \succeq (g, [\mathcal{P}_2, c_2])$$

if, and only if,

$$\begin{array}{c} \alpha \underset{p \in co(\mathcal{P}_1)}{min} \int u \circ f dp + (1-\alpha) \int u \circ f dc_1 = \underset{p \in co(\mathcal{P}^{\alpha})}{min} \int u \circ f dp \geq \\ \alpha \underset{p \in co(\mathcal{P}_2)}{min} \int u \circ g dp + (1-\alpha) \int u \circ g dc_2 = \underset{p \in co(\mathcal{P}^{\alpha})}{min} \int u \circ g dp \end{array}$$

The parameter α in the above representation theorem can be interpreted as a degree of pessimism of the decision maker. If $\alpha = 0$, he behaves as an expected utility maximizer with respect to the anchor probability, while if $\alpha = 1$, he behaves as a maximizer of the minimum expected utility with respect to *all* the distributions compatible with his information.

4.4 Comparative imprecision aversion

We end this section by showing that a decision maker is more averse towards imprecision than another if, in any given situation, his revealed set of priors is included in the second decision maker's revealed set. Assume from now on that the set of consequences X is equal to $[0, M] \subset \mathbb{R}$ and assume that preferences respect the natural order on $[0, M]$, i.e., if $x, y \in [0, M]$ and $x > y$, then, in any possible situation, the decision maker prefers getting the constant degenerate lottery k_{δ_x} giving him x in all the states to getting k_{δ_y}.

Let Y_B be the set of lotteries over $\{0, M\}$, i.e., the set of lotteries whose outcome consist only of the two extreme prizes. $y \in Y_B$ can be written $(0, p; M, 1-p)$. Similarly, let $\mathcal{A}_B$ be the set of acts defined on Y_B, i.e., acts that can be written $f(s) = y_s$ with $y_s = (0, 1-p_s; M, p_s) \in Y_B$ for all $s \in S$.

Definition 7 *Let $\succeq_a$ and $\succeq_b$ be two preference relations defined on $\mathcal{A} \times \mathcal{S}$. We say that $\succeq_b$ is more imprecision averse than $\succeq_a$ if for all $y \in Y_B$ and all $f \in \mathcal{A}_B$, and for all $[\mathcal{P}, c] \in \mathcal{S}$,*

$$(k_y, [\mathcal{P}, c]) \succeq_a (f, [\mathcal{P}, c]) \Rightarrow (k_y, [\mathcal{P}, c]) \succeq_b (f, [\mathcal{P}, c])$$

and

$$(k_y, [\mathcal{P}, c]) \succ_a (f, [\mathcal{P}, c]) \Rightarrow (k_y, [\mathcal{P}, c]) \succ_b (f, [\mathcal{P}, c])$$

where k_y is the constant act giving the binary lottery y in all states.

Note that this definition differs from the one in Ghirardato and Marinacci (2002) and Epstein (1999), in that we restrict attention to comparisons among binary lotteries and binary acts.

Definition 8 *Let $(f, [\mathcal{P}, c]) \in \mathcal{A} \times \mathcal{S}$. Call the* probabilistic binary equivalent *of $(f, [\mathcal{P}, c])$ the lottery $Pe(f, [\mathcal{P}, c]) = (0, 1-p; M, p) \in Y_B$ such that $(f, [\mathcal{P}, c]) \sim (k_{Pe(f,[\mathcal{P},c])}, [\mathcal{P}, c])$, where $k_{Pe(f,[\mathcal{P},c])}$ is the constant act giving $Pe(f, [\mathcal{P}, c])$.*

Note that, under continuity, such a probabilistic binary equivalent always exist. Probabilistic binary equivalent can be associated to the probability p of getting M in the lottery $Pe(f, [\mathcal{P}, c])$. Denote $\Pi(f, [\mathcal{P}, c])$ this quantity, i.e., $Pe(f, [\mathcal{P}, c]) = (0, 1 - \Pi(f, [\mathcal{P}, c]); M, \Pi(f, [\mathcal{P}, c]))$

Normalize utilities so that $u(\delta_0) = 0$ and $u(\delta_M) = 1$.

Proposition 1 *$\succeq$ satisfy all axioms of theorem 2. If $[\mathcal{P}_1, c_1]$ is a center preserving imprecision increase of $[\mathcal{P}_2, c_2]$, then, for all $f \in \mathcal{A}$, $\Pi(f, [\mathcal{P}_2, c_2]) \leq \Pi(f, [\mathcal{P}_1, c_1])$.*

Proof. Straightforward. ■

Theorem 4 *Let $\succeq_a$ and $\succeq_b$ be two preference relations defined on $\mathcal{A} \times \mathcal{S}$, satisfying all axioms of theorem 2. Then, the following assertions are equivalent*

(i) $\succeq_b$ is more averse towards imprecision than $\succeq_a$

(ii) for all $(f, [\mathcal{P}, c]) \in \mathcal{A}_B \times \mathcal{S}$, $\Pi_a(f, [\mathcal{P}, c]) \geq \Pi_b(f, [\mathcal{P}, c])$

(iii) for all $[\mathcal{P}, c] \in \mathcal{S}$, $\mathcal{F}^a_{[\mathcal{P},c]} \subseteq \mathcal{F}^b_{[\mathcal{P},c]}$

This theorem provides an easy way of comparing two decision makers in terms of their attitudes towards imprecision: the decision maker operating the largest contraction on the set of compatible priors is the one who is the less averse towards imprecision.

5 Discussion and concluding remarks

We have axiomatized a decision criterion that links the completely subjective set of priors revealed by choice behavior in Gilboa and Schmeidler (1989) to the available information. Our approach is based on the idea that the latter can be represented through a set of priors together with a reference prior, as in Handsen and Sargent (2002) and Wang (2001). However, contrary to Wang (2001), our main axiom of aversion towards information imprecision is sufficiently weak to enable us to model a wide array of pessimistic behavior, from full pessimism (maxmin over the entire set of priors) to imprecision neutrality (expected utility with respect to the reference prior). This approach, based on a description of information independent of the choice behavior of the decision maker, parallels the usual approach to risk, in which risk aversion is based on second order stochastic dominance. Our hope is that it will provide a useful benchmark in economics to pursue comparative statics exercise, in which the precision of the information is changed. Hence, the framework seems susceptible to be applied to study a wide range of question, as the impact of information campaign on choices, the reaction of markets after say, central bank announcements, ... Another potential source of application is to model scientific uncertainty, for instance to provide guidance in environmental policies or in issues like new sanitary "risks" (like the mad cow disease for instance).

Much remains to be done to have a full fledge theory of reaction to different pieces of (objective) information. In this paper, we chose to model information through a set of priors together with a reference prior. This in particular encompasses situations in which a second order probability distribution is given (as in Segal (1987)). However, this is clearly not the only way to model prior information. In ongoing research, we drop the idea of a reference prior to model the more general case in which the only given is a set of priors.

Appendix

Proof. [Lemma 1] Let $\alpha \in]0,1]$. Since $c \in co(\mathcal{P})$, $\alpha p + (1-\alpha)c \in co(\mathcal{P})$ for all $p \in \mathcal{P}$, thus proving that $co(\mathcal{P}) \supseteq co(\mathcal{P}^{\alpha})$, i.e., that condition (1) of Definition 2 is satisfied.
Condition (2) is obviously satisfied. ■

Proof. [Lemma 2] Consider φ_i defined for $i = 1, 2$ by $\varphi_i(s) = 1$ for $s \in \{1, \max(S(\mathcal{P}_i))\}$, $\varphi_i(s) = s + 1 - \max(S(\mathcal{P}_i))$ for $s > \max(S(\mathcal{P}_i))$. Since $S(\mathcal{P}_i)$ is a finite set, φ_i is an onto mapping. Remark then that f is trivially φ_i-measurable for $i = 1, 2$ since it is a constant act, $\mathcal{P}_1^{\varphi_1} = c_1^{\varphi_1} = \mathcal{P}_2^{\varphi_2} = c_2^{\varphi_2}$ and $f^{\varphi_1} = f^{\varphi_2}$ and thus $(f, [\mathcal{P}_1, c_1])$ and $(f, [\mathcal{P}_2, c_2])$ are equivalent. ■

Proof. [Theorem 1] $(\Rightarrow)$

Building on Gilboa and Schmeidler (1989)'s representation result, we know there exist $\mathcal{F}_{[\mathcal{P}_i,c_i]}$ as well as $u_{[\mathcal{P}_i,c_i]}$ such that $(f, [\mathcal{P}_i, c_i]) \succeq (g, [\mathcal{P}_i, c_i])$ iff:

$$\min_{p \in \mathcal{F}_{[\mathcal{P}_i,c_i]}} \int u_{[\mathcal{P}_i,c_i]} \circ f dp \geq \min_{p \in \mathcal{F}_{[\mathcal{P}_i,c_i]}} \int u_{[\mathcal{P}_i,c_i]} \circ g dp.$$

Hence, we need to show first that the utility function can be taken independently of the situation and, second, that the representation can be extended to acts associated with different situations.

Let $[\mathcal{P}_i, c_i]$ and $[\mathcal{P}_j, c_j]$ be two given situations in $\mathcal{S}$. We know that the decision maker is Expected Utility Maximizer over constant acts. Lemma 2 implies that $u_{[\mathcal{P}_i,c_i]}$ and $u_{[\mathcal{P}_j,c_j]}$ represent the same expected utility over constant acts (which implies that $\succeq^{\ell}_{[\mathcal{P}_i,c_i]} = \succeq^{\ell}_{[\mathcal{P}_j,c_j]} = \succeq^{\ell}$). Hence, they can be taken to be equal, i.e., $u_{[\mathcal{P}_i,c_i]} = u_{[\mathcal{P}_j,c_j]} = u$.

To show that the representation can be extended to compare acts associated to different contexts, let $(f, [\mathcal{P}_i, c_i]) \succeq (g, [\mathcal{P}_j, c_j])$. Since $S(\mathcal{P}_i)$ and $S(\mathcal{P}_j)$ are finite and $f(s)$ and $g(s)$ have finite support, there exist $\overline{x}$ and $\underline{x}$ in X such that for all $s \in S(\mathcal{P}_i) \cup S(\mathcal{P}_j)$, for all $x \in Supp(f(s)) \cup Supp(g(s))$, $\delta_{\overline{x}} \succeq^{\ell} \delta_x \succeq^{\ell} \delta_{\underline{x}}$. Hence, by axioms 4 and 8 we know that $(k_{\overline{x}}, [\mathcal{P}_i, c_i]) \succeq (f, [\mathcal{P}_i, c_i]) \succeq \left(k_{\underline{x}}, [\mathcal{P}_i, c_i]\right)$ and $(k_{\overline{x}}, [\mathcal{P}_j, c_j]) \succeq (g, [\mathcal{P}_j, c_j]) \succeq \left(k_{\underline{x}}, [\mathcal{P}_j, c_j]\right)$ where $k_{\overline{x}}$ (resp. $k_{\underline{x}}$) is the constant act giving $\delta_{\overline{x}}$ (resp. $\delta_{\underline{x}}$) in all states. By axioms 1 and 3, there exists λ_i such that $(f, [\mathcal{P}_i, c_i]) \sim (\lambda_i k_{\overline{x}} + (1-\lambda_i)k_{\underline{x}}, [\mathcal{P}_i, c_i])$. Similarly, there exists λ_j such that $(g, [\mathcal{P}_j, c_j]) \sim (\lambda_j k_{\overline{x}} + (1-\lambda_j)k_{\underline{x}}, [\mathcal{P}_j, c_j])$. Thus,

$$\begin{aligned}(f, [\mathcal{P}_i, c_i]) \succeq (g, [\mathcal{P}_j, c_j]) \quad &\Leftrightarrow \quad (\lambda_i k_{\overline{x}} + (1-\lambda_i)k_{\underline{x}}, [\mathcal{P}_i, c_i]) \succeq (\lambda_j k_{\overline{x}} + (1-\lambda_j)k_{\underline{x}}, [\mathcal{P}_j, c_j]) \\ &\Leftrightarrow \quad \lambda_i k_{\overline{x}} + (1-\lambda_i)k_{\underline{x}} \succeq^{\ell} \lambda_j k_{\overline{x}} + (1-\lambda_j)k_{\underline{x}}\end{aligned}$$

Now, $(f, [\mathcal{P}_i, c_i]) \sim (\lambda_i k_{\overline{x}} + (1-\lambda_i)k_{\underline{x}}, [\mathcal{P}_i, c_i])$ implies that $\min_{p \in \mathcal{F}_{[\mathcal{P}_i,c_i]}} \int u \circ f dp = u(\lambda_i \delta_{\overline{x}} + (1-\lambda_i)\delta_{\underline{x}})$. We also have that $\min_{p \in \mathcal{F}_{[\mathcal{P}_j,c_j]}} \int u \circ g dp = u(\lambda_j \delta_{\overline{x}} + (1-\lambda_j)\delta_{\underline{x}})$ and $u(\lambda_i \delta_{\overline{x}} + (1-\lambda_i)\delta_{\underline{x}}) \geq u(\lambda_j \delta_{\overline{x}} + (1-\lambda_j)\delta_{\underline{x}})$, which implies that

$$\min_{p \in \mathcal{F}_{[\mathcal{P}_i,c_i]}} \int u \circ f dp \geq \min_{p \in \mathcal{F}_{[\mathcal{P}_j,c_j]}} \int u \circ g dp$$

To prove condition 1, consider $p^* \in \mathcal{F}_{[\mathcal{P},c]}$ and suppose that $p^*(S(\mathcal{P})) = q \neq 1$. Consider $\overline{x}$ and $\underline{x}$ in X such that $u(\delta_{\overline{x}}) > u(\delta_{\underline{x}})$ and let f be defined by $f(s) = \delta_{\overline{x}}$ for all $s \in S(\mathcal{P})$, $f(s) = \delta_{\underline{x}}$ for all $s \in S \backslash S(\mathcal{P})$, and g by $g(s) = \delta_{\overline{x}}$ for all $s \in S$. Then

$$\min_{p \in \mathcal{F}_{[\mathcal{P},c]}} \int u \circ f dp \leq \int u \circ f dp^* = qu(\overline{x}) + (1-q)u(\underline{x}) < u(\overline{x}) = \min_{p \in \mathcal{F}_{[\mathcal{P},c]}} \int u \circ g dp$$

and thus $(g, [\mathcal{P}, c]) \succ (f, [\mathcal{P}, c])$ which is a violation of axiom 8 since $g = f_{S(\mathcal{P})}g$. Thus, for all $p \in \mathcal{F}_{[\mathcal{P},c]}$, $p(S(\mathcal{P})) = 1$.

To prove condition 2, consider φ an onto mapping from S to S, and $[\mathcal{P}, c] \in \mathcal{S}$. Suppose first that $\mathcal{F}_{[\mathcal{P},c]^\varphi} \not\subseteq \{p^\varphi | p \in \mathcal{F}_{[\mathcal{P},c]}\}$ and that there exists $p^* \in \mathcal{F}_{[\mathcal{P}^\varphi,c^\varphi]}$ such that $p^* \notin \{p^\varphi | p \in \mathcal{F}_{[\mathcal{P},c]}\}$. Observe that since $\mathcal{F}_{[\mathcal{P},c]}$ is a convex set, $\left(\mathcal{F}_{[\mathcal{P},c]}\right)^\varphi = \{p^\varphi | p \in \mathcal{F}_{[\mathcal{P},c]}\}$ is also convex since by definition of p^φ, for all $\alpha \in [0,1]$, for all $p_1, p_2 \in \Delta(S)$, $(\alpha p_1 + (1-\alpha)p_2)^\varphi = \alpha p_1^\varphi + (1-\alpha)p_2^\varphi$.

Hence, using a separation argument, we know there exists a function $\phi : S \to \mathbb{R}$ such that $\int \phi dp^* < \min_{p \in \left(\mathcal{F}_{[\mathcal{P},c]}\right)^\varphi} \int \phi dp$. Since $S(\mathcal{P}^\varphi)$ is a finite set, there exist numbers a, b with $a > 0$, such that $\forall s \in S(\mathcal{P}^\varphi)$, $(a\phi(s) + b) \in u(Y)$. Then, for all $s \in S(\mathcal{P}^\varphi)$ there exists $y(s) \in Y$ such that $u(y(s)) = a\phi(s) + b$. Define f by $f(s) = y(s)$ for all $s \in S(\mathcal{P}^\varphi)$, $f(s) = \delta_x$ for all $s \in S \backslash S(\mathcal{P}^\varphi)$, where $x \in X$. Define g by $g(s) = y(\varphi(s))$ for all $s \in S$, that is $f = g^\varphi$. Thus $(g, [\mathcal{P}, c])$ is equivalent to $(f, [\mathcal{P}^\varphi, c^\varphi])$ and we have that for all $p \in \mathcal{F}_{[\mathcal{P},c]}$, $\int u \circ g dp = \int u \circ g^\varphi dp^\varphi$ and thus

$$\begin{aligned} \min_{p \in \mathcal{F}_{[\mathcal{P},c]}} \int u \circ g dp &= \min_{p \in \left(\mathcal{F}_{[\mathcal{P},c]}\right)^\varphi} \int u \circ g^\varphi dp = \min_{p \in \left(\mathcal{F}_{[\mathcal{P},c]}\right)^\varphi} \int u \circ f dp \\ &= \min_{p \in \left(\mathcal{F}_{[\mathcal{P},c]}\right)^\varphi} \int (a\phi + b) dp > \int (a\phi + b) dp^* \geq \min_{p \in \mathcal{F}_{[\mathcal{P}^\varphi,c^\varphi]}} \int u \circ f dp \end{aligned}$$

which shows that $(g, [\mathcal{P}, c]) \succ (f, [\mathcal{P}^\varphi, c^\varphi])$ which is a violation of axiom 7.

Suppose now that $\mathcal{F}_{[\mathcal{P}^\varphi,c^\varphi]} \not\supseteq \{p^\varphi | p \in \mathcal{F}_{[\mathcal{P},c]}\}$ and that there exists $p^* \in \{p^\varphi | p \in \mathcal{F}_{[\mathcal{P},c]}\}$ such that $p^* \notin \mathcal{F}_{[\mathcal{P}^\varphi,c^\varphi]}$. Using a separation argument, we know there exists a function $\phi : S \to \mathbb{R}$ such that $\int \phi dp^* < \min_{p \in \mathcal{F}_{[\mathcal{P}^\varphi,c^\varphi]}} \int \phi dp$. Since $S(\mathcal{P}^\varphi)$ is a finite set, there exist numbers a, b with $a > 0$, such that $\forall s \in S(\mathcal{P}^\varphi)$, $(a\phi(s) + b) \in u(Y)$. Then, for all $s \in S(\mathcal{P}^\varphi)$ there exists $y(s) \in Y$ such that $u(y(s)) = a\phi(s) + b$. Define f by $f(s) = y(s)$ for all $s \in S(\mathcal{P}^\varphi)$, $f(s) = \delta_x$ for all $s \in S \backslash S(\mathcal{P}^\varphi)$, where $x \in X$. Define g by $g(s) = y(\varphi(s))$ for all $s \in S$, that is $f = g^\varphi$. Thus $(g, [\mathcal{P}, c])$ is equivalent to $(f, [\mathcal{P}^\varphi, c^\varphi])$ and we have that for all $p \in \mathcal{F}_{[\mathcal{P},c]}$, $\int u \circ g dp = \int u \circ g^\varphi dp^\varphi$ and thus

$$\begin{aligned} \min_{p \in \mathcal{F}_{[\mathcal{P},c]}} \int u \circ g dp &= \min_{p \in \left(\mathcal{F}_{[\mathcal{P},c]}\right)^\varphi} \int u \circ g^\varphi dp = \min_{p \in \left(\mathcal{F}_{[\mathcal{P},c]}\right)^\varphi} \int u \circ f dp \\ &= \min_{p \in \left(\mathcal{F}_{[\mathcal{P},c]}\right)^\varphi} \int (a\phi + b) dp \leq \int (a\phi + b) dp^* < \min_{p \in \mathcal{F}_{[\mathcal{P}^\varphi,c^\varphi]}} \int u \circ f dp \end{aligned}$$

which shows that $(g, [\mathcal{P}, c]) \prec (f, [\mathcal{P}^\varphi, c^\varphi])$ which is a violation of axiom 7.

($\Leftarrow$) The only axioms besides Gilboa and Schmeidler's are axioms 7 and 8.

Consider $(f, [\mathcal{P}_i, c_i])$ and $(g, [\mathcal{P}_j, c_j])$ which are equivalent, that is, there exists φ_i and φ_j two onto mappings from S to S, such that f is φ_i-measurable, g is φ_j-measurable, $\mathcal{P}_i^{\varphi_i} = \mathcal{P}_j^{\varphi_j}$, $c^{\varphi_i} = c^{\varphi_j}$ and $f^{\varphi_i} = g^{\varphi_j}$. Remark that for all $p \in \mathcal{F}_{[\mathcal{P}_i, c_i]}$, $\int u \circ f dp = \int u \circ f^{\varphi_i} dp^{\varphi_i}$ and thus

$$\min_{p \in \mathcal{F}_{[\mathcal{P}_i, c_i]}} \int u \circ f dp = \min_{p \in \left(\mathcal{F}_{[\mathcal{P}_i, c_i]}\right)^{\varphi_i}} \int u \circ f^{\varphi_i} dp$$

Observe also that for all $p \in \mathcal{F}_{[\mathcal{P}_j, c_j]}$, $\int u \circ g dp = \int u \circ g^{\varphi_j} dp^{\varphi_j}$ and thus

$$\min_{p \in \mathcal{F}_{[\mathcal{P}_j, c_j]}} \int u \circ g dp = \min_{p \in \left(\mathcal{F}_{[\mathcal{P}_j, c_j]}\right)^{\varphi_j}} \int u \circ g^{\varphi_j} dp$$

Since condition 2 holds, we have $\mathcal{F}_{[\mathcal{P}_i^{\varphi_i}, c_i^{\varphi_i}]} = \left(\mathcal{F}_{[\mathcal{P}_i, c_i]}\right)^{\varphi_i}$ and $\mathcal{F}_{\left[\mathcal{P}_j^{\varphi_j}, c_j^{\varphi_j}\right]} = \left(\mathcal{F}_{[\mathcal{P}_j, c_j]}\right)^{\varphi_j}$. Since by hypothesis $[\mathcal{P}_i^{\varphi_i}, c_i^{\varphi_i}] = \left[\mathcal{P}_j^{\varphi_j}, c_j^{\varphi_j}\right]$ and $f^{\varphi_i} = g^{\varphi_j}$ we have that $\min_{p \in \mathcal{F}_{[\mathcal{P}_i, c_i]}} \int u \circ f dp = \min_{p \in \mathcal{F}_{[\mathcal{P}_j, c_j]}} \int u \circ g dp$ and thus $(f, [\mathcal{P}_i, c_i]) \sim (g, [\mathcal{P}_j, c_j])$ which shows that axiom 7 holds.

Condition 1 obviously implies axiom 8. ∎

Proof. [Lemma 3] Let $[\mathcal{P}, c] \in \mathcal{S}$. By definition,

$$c_R^{(\alpha,\varphi)} = \alpha c + (1-\alpha) c^{\varphi} = c_M^{(\alpha,\varphi)}$$

Condition (1) of Definition 2 is also satisfied: for all $q \in \mathcal{P}_R^{(\alpha,\varphi)})$, there exists a (unique) $p \in \mathcal{P}$ such that $q = \alpha p + (1-\alpha) p^{\varphi}$. Hence, $q \in \mathcal{P}_M^{(\alpha,\varphi)}$ and therefore $\mathcal{P}_R^{(\alpha,\varphi)} \subset \mathcal{P}_M^{(\alpha,\varphi)}$. ∎

Proof. [Lemma 4] Let $(f, [\mathcal{P}, c])$ and $(g, [\mathcal{P}, c])$ be given, such that $(f, [\mathcal{P}, c]) \sim (g, [\mathcal{P}, c])$. Without loss of generality, let $S(\mathcal{P}) = \{1, ..., n\}$.

Consider the bijection $\varphi : S \to S$ such that $\varphi(s) = s + n$ for all $s \in \{1, ..., n\}$, $\varphi(s) = s - n$ for all $s \in \{n+1, ..., 2n\}$ and $\varphi(s) = s$ for all $s > 2n$.

Consider now the onto mapping $\psi : S \to S$ such that $\psi(s) = s$ for all $s \in \{1, ..., n\}$, $\psi(s) = s - n$ for all $s > n$ and the act $g_{S(\mathcal{P})} g^{\varphi}$: $g_{S(\mathcal{P})} g^{\varphi}$ is ψ-measurable and thus $(g_{S(\mathcal{P})} g^{\varphi}, [\mathcal{P}^{\varphi}, c^{\varphi}])$ is equivalent to $\left(\left(g_{S(\mathcal{P})} g^{\varphi}\right)^{\psi}, [(\mathcal{P}^{\varphi})^{\psi}, (c^{\varphi})^{\psi}]\right)$. By axiom 7,

$$(g_{S(\mathcal{P})} g^{\varphi}, [\mathcal{P}^{\varphi}, c^{\varphi}]) \sim \left(\left(g_{S(\mathcal{P})} g^{\varphi}\right)^{\psi}, \left[(\mathcal{P}^{\varphi})^{\psi}, (c^{\varphi})^{\psi}\right]\right)$$

Observe that $\left[(\mathcal{P}^{\varphi})^{\psi}, (c^{\varphi})^{\psi}\right] = [\mathcal{P}, c]$ and for all $s \in \{1, ..., n\}$, $\left(g_{S(\mathcal{P})} g^{\varphi}\right)^{\psi}(s) = g(s)$. Thus, by axiom 8, we have that

$$(g, [\mathcal{P}, c]) \sim \left(\left(g_{S(\mathcal{P})} g^{\varphi}\right)^{\psi}, \left[(\mathcal{P}^{\varphi})^{\psi}, (c^{\varphi})^{\psi}\right]\right) \sim \left(g_{S(\mathcal{P})} g^{\varphi}, [\mathcal{P}^{\varphi}, c^{\varphi}]\right)$$

Since $S(\mathcal{P}\varphi) \cap S(\mathcal{P}) = \emptyset$, by axiom 8, we also have

$$(f, [\mathcal{P}, c]) \sim (f_{S(\mathcal{P})} g^{\varphi}, [\mathcal{P}, c]) \text{ and } (g_{S(\mathcal{P})} g^{\varphi}, [\mathcal{P}^{\varphi}, c^{\varphi}]) \sim (f_{S(\mathcal{P})} g^{\varphi}, [\mathcal{P}^{\varphi}, c^{\varphi}])$$

Thus, since $(f_{S(\mathcal{P})}g^{\varphi}, [\mathcal{P}, c]) \sim (g_{S(\mathcal{P})}g^{\varphi}, [\mathcal{P}, c])$ and $(f_{S(\mathcal{P})}g^{\varphi}, [\mathcal{P}^{\varphi}, c^{\varphi}]) \sim (g_{S(\mathcal{P})}g^{\varphi}, [\mathcal{P}^{\varphi}, c^{\varphi}])$, by axiom 10, we have that

$$\left(f_{S(\mathcal{P})}g^{\varphi}, \left[\mathcal{P}_M^{(\alpha,\varphi)}, c_M^{(\alpha,\varphi)}\right]\right) \sim \left(g_{S(\mathcal{P})}g^{\varphi}, \left[\mathcal{P}_M^{(\alpha,\varphi)}, c_M^{(\alpha,\varphi)}\right]\right)$$

Consider again the onto mapping ψ. By axiom 7,

$$\left(g_{S(\mathcal{P})}g^{\varphi}, \left[\mathcal{P}_M^{(\alpha,\varphi)}, c_M^{(\alpha,\varphi)}\right]\right) \sim \left(\left(g_{S(\mathcal{P})}g^{\varphi}\right)^{\psi}, \left[(\mathcal{P}_M^{(\alpha,\varphi)})^{\psi}, (c_M^{(\alpha,\varphi)})^{\psi}\right]\right)$$

Observe that $\left[(\mathcal{P}_M^{(\alpha,\varphi)})^{\psi}, (c_M^{(\alpha,\varphi)})^{\psi}\right] = [\mathcal{P}, c]$ and for all $s \in \{1, ..., n\}$, $\left(g_{S(\mathcal{P})}g^{\varphi}\right)^{\psi}(s) = g(s)$. Thus, by axiom 8, we have that

$$(g, [\mathcal{P}, c]) \sim \left(\left(g_{S(\mathcal{P})}g^{\varphi}\right)^{\psi}, \left[(\mathcal{P}_M^{(\alpha,\varphi)})^{\psi}, (c_M^{(\alpha,\varphi)})^{\psi}\right]\right) \sim \left(g_{S(\mathcal{P})}g^{\varphi}, \left[\mathcal{P}_M^{(\alpha,\varphi)}, c_M^{(\alpha,\varphi)}\right]\right)$$

and hence, we get that $(f, [\mathcal{P}, c]) \sim (g, [\mathcal{P}, c]) \sim \left(f_{S(\mathcal{P})}g^{\varphi}, \left[\mathcal{P}_M^{(\alpha,\varphi)}, c_M^{(\alpha,\varphi)}\right]\right)$. ■

Proof. [Lemma 5] We first check axiom 2. Let $(f, [\mathcal{P}, c]) \succ (g, [\mathcal{P}, c])$ and $h \in \mathcal{A}^c$ (i.e., h is a constant act). By axiom 7, $(\alpha f + (1-\alpha)h, [\mathcal{P}, c]) \sim (\alpha f_{S(\mathcal{P})}h + (1-\alpha)h, [\mathcal{P}c])$, and hence by axiom 11,

$$(\alpha f + (1-\alpha)h, [\mathcal{P}, c]) \sim \left(f_{S(\mathcal{P})}h, \left[\mathcal{P}_R^{(\alpha,\varphi)}, c_R^{(\alpha,\varphi)}\right]\right)$$

Define $\phi : S \to S$ such that $\phi(S(\mathcal{P})) = S(\mathcal{P})$ and $\phi \circ \varphi(S(\mathcal{P})) = \{s^{\star}\}$ where $s^{\star} \in S \setminus S(\mathcal{P})$. By axiom 7,

$$\left(f_{S(\mathcal{P})}h, \left[\mathcal{P}_R^{(\alpha,\varphi)}, c_R^{(\alpha,\varphi)}\right]\right) \sim \left(f_{S(\mathcal{P})}h^{\varphi})^{\phi}, \left[\left(\mathcal{P}_R^{(\alpha,\varphi)}\right)^{\phi}, \left(c_R^{(\alpha,\varphi)}\right)^{\phi}\right]\right)$$

Now recall the that $\mathcal{P}_R^{(\alpha,\varphi)} = \{q | q = \alpha p + (1-\alpha)p^{\varphi}, p \in P)\}$ and hence

$$\left(\mathcal{P}_R^{(\alpha,\varphi)}\right)^{\phi} = \left\{r | r = \alpha p + (1-\alpha)\delta_{\phi\circ\varphi(S\mathcal{P})}, p \in P)\right\}$$

Thus, this set is actually the mixture of two underlying sets and hence:

$$\left(\left(f_{S(\mathcal{P})}h^{\varphi}\right)^{\phi}, \left[\left(\mathcal{P}_R^{(\alpha,\varphi)}\right)^{\phi}, \left(c_R^{(\alpha,\varphi)}\right)^{\phi}\right]\right) = \left(\left(f_{S(\mathcal{P})}h^{\varphi}\right)^{\phi}, \alpha[\mathcal{P}, c] + (1-\alpha)\left[\{\delta_{\phi\circ\varphi(\mathcal{P})}\}, \delta_{\phi\circ\varphi(\mathcal{P})}\right]\right)$$

Hence, we have so far established that

$$(\alpha f + (1-\alpha)h, [\mathcal{P}, c]) \sim \left(\left(f_{S(\mathcal{P})}h^{\varphi}\right)^{\phi}, \alpha[\mathcal{P}, c] + (1-\alpha)\left[\{\delta_{\phi\circ\varphi(\mathcal{P})}\}, \delta_{\phi\circ\varphi(\mathcal{P})}\right]\right)$$

A similar statement is true working with g instead of h.
Now, by axiom 7 and by construction, we have:

$$\left(\left(f_{S(\mathcal{P})}h^{\varphi}\right)^{\phi}, [\mathcal{P}, c]\right) \sim (f, [\mathcal{P}, c]) \succ (g, [\mathcal{P}, c]) \sim \left(\left(g_{S(\mathcal{P})}h^{\varphi}\right)^{\phi}, [\mathcal{P}, c]\right)$$

and

$$\left(\left(f_{S(\mathcal{P})}h^{\varphi}\right)^{\phi}, \left[\{\delta_{\phi\circ\varphi(\mathcal{P})}\}, \delta_{\phi\circ\varphi(\mathcal{P})}\right]\right) \sim \left(\left(g_{S(\mathcal{P})}h^{\varphi}\right)^{\phi}, \left[\{\delta_{\phi\circ\varphi(\mathcal{P})}\}, \delta_{\phi\circ\varphi(\mathcal{P})}\right]\right)$$

Hence, axiom 10 implies that

$$\left(\left(f_{S(\mathcal{P})}h^{\varphi}\right)^{\phi}, \alpha[\mathcal{P}, c] + (1-\alpha)\left[\{\delta_{\phi\circ\varphi(\mathcal{P})}\}, \delta_{\phi\circ\varphi(\mathcal{P})}\right]\right) \succeq \left(\left(g_{S(\mathcal{P})}h^{\varphi}\right)^{\phi}, \alpha[\mathcal{P}, c] + (1-\alpha)\left[\{\delta_{\phi\circ\varphi(\mathcal{P})}\}, \delta_{\phi\circ\varphi(\mathcal{P})}\right]\right)$$

which in turn implies that

$$(\alpha f + (1-\alpha)h, [\mathcal{P}, c]) \succeq (\alpha g + (1-\alpha)h, [\mathcal{P}, c])$$

thus proving axiom 2.

We now check that 6 holds as well. Let $(f, [\mathcal{P}, c])$ and $(g, [\mathcal{P}, c])$ be given, such that $(f, [\mathcal{P}, c]) \sim (g, [\mathcal{P}, c])$. According to lemma 4, we have that

$$(f, [\mathcal{P}, c]) \sim (g, [\mathcal{P}, c]) \sim \left(f_{S(\mathcal{P})}g^{\varphi}, \left[\mathcal{P}_M^{(\alpha,\varphi)}, c_M^{(\alpha,\varphi)}\right]\right)$$

In lemma 3, it was shown that $\left[\mathcal{P}_M^{(\alpha,\varphi)}, c_M^{(\alpha,\varphi)}\right]$ was more imprecise than $\left[\mathcal{P}_R^{(\alpha,\varphi)}, c_R^{(\alpha,\varphi)}\right]$. Thus axiom 9 implies that

$$\left(f_{S(\mathcal{P})}g^{\varphi}, \left[\mathcal{P}_R^{(\alpha,\varphi)}, c_R^{(\alpha,\varphi)}\right]\right) \succeq \left(f_{S(\mathcal{P})}g^{\varphi}, \left[\mathcal{P}_M^{(\alpha,\varphi)}, c_M^{(\alpha,\varphi)}\right]\right)$$

Since by axiom 11, we have that $(\alpha f + (1-\alpha)g, [\mathcal{P}, c]) \sim \left(f_{S(\mathcal{P})}g^{\varphi}, \left[\mathcal{P}_R^{(\alpha,\varphi)}, c_R^{(\alpha,\varphi)}\right]\right)$, it follows that

$$(\alpha f + (1-\alpha)g, [\mathcal{P}, c]) \succeq (f, [\mathcal{P}, c]) \sim (g, [\mathcal{P}, c])$$

Hence, axiom 6 is satisfied. ■

Proof. **[Lemma 6]** Let consider $f, g \in \mathcal{A}$, $[\mathcal{P}, c] \in \mathcal{S}$. Then for all $p \in \Delta(S)$ such that $p \in \mathcal{P}$, we have $(f, [\{p\}, p]) \sim (f_{S(\mathcal{P})}g, [\{p\}, p])$. Thus, by axiom 12, $(f, [\mathcal{P}, c]) \sim (f_{S(\mathcal{P})}g, [\mathcal{P}, c])$. ■

Proof. **[Theorem 2]** $(\Rightarrow)$ By lemma 5, we know that axioms 7 to 11 imply axioms 2 and 6. Furthermore axiom 12 imply axiom 8 (lemma 7). Hence, we can invoke theorem 1, to prove that there exists an unique (up to a positive linear transformation) affine function $u : Y \to \mathbb{R}$, and for all $[\mathcal{P}_i, c_i], [\mathcal{P}_j, c_j] \in \mathcal{S}$, there exist unique, non-empty, closed and convex set $\mathcal{F}_{[\mathcal{P}_i, c_i]}$ and $\mathcal{F}_{[\mathcal{P}_j, c_j]}$ of finitely additive probability measures on 2^S, such that for all $f, g \in \mathcal{A}$, $(f, [\mathcal{P}_i, c_i]) \succeq (g, [\mathcal{P}_j, c_j])$ if, and only if:

$$\min_{p \in \mathcal{F}_{[\mathcal{P}_i, c_i]}} \int u \circ f dp \geq \min_{p \in \mathcal{F}_{[\mathcal{P}_j, c_j]}} \int u \circ g dp$$

Furthermore, for all φ onto mapping from S to S, $\mathcal{F}_{[\mathcal{P}_i^{\varphi}, c_i^{\varphi}]} = \left\{p^{\varphi} | p \in \mathcal{F}_{[\mathcal{P}_i, c_i]}\right\}$.

Suppose condition 1 does not hold, that is there exists $[\mathcal{P},c] \in \mathcal{S}$ such that $\mathcal{F}_{[\mathcal{P},c]} \nsubseteq \overline{co}(\mathcal{P}))$. Hence, there exists $p^* \in \mathcal{F}_{[\mathcal{P},c]}$ such that $p^* \notin \overline{co}(\mathcal{P})$. Since $\overline{co}(\mathcal{P})$ is a convex set, using a separation argument, we know there exists a function $\phi : S \rightarrow \mathbb{R}$ such that $\int \phi dp^* < \min_{p\in\overline{co}(\mathcal{P})} \int \phi dp$. Note that since axiom 12 implies axiom 8, we have that condition 1 in theorem 1 holds and thus, for all $p \in \mathcal{F}_{[\mathcal{P},c]}$, $p(S(\mathcal{P})) = 1$. Thus $Supp(p^*) \subseteq S(\mathcal{P})$ and since $S(\mathcal{P})$ is a finite set, there exist numbers a, b with $a > 0$, such that $\forall s \in S(\mathcal{P})$, $(a\phi(s) + b) \in u(Y)$. Then, for all $s \in S(\mathcal{P})$ there exists $y(s) \in Y$ such that $u(y(s)) = a\phi(s) + b$. Define f by $f(s) = y(s)$ for all $s \in S(\mathcal{P})$, $f(s) = \delta_x$ for all $s \in S\backslash S(\mathcal{P})$ where $x \in X$. Note that $\min_{p\in\overline{co}(\mathcal{P})} \int (a\phi + b)dp \in Y$ and thus there exists y^* such that $u(y^*) = \min_{p\in\overline{co}(\mathcal{P})} \int (a\phi + b)dp$. Define g by $g(s) = y^*$ for all $s \in S$. Observe that for all $p \in \Delta(S)$ such that $p \in \mathcal{P}$, $p \in \overline{co}(\mathcal{P})$ and thus

$$\int u \circ f dp \geq \min_{p\in\overline{co}(\mathcal{P})} \int u \circ f dp = \min_{p\in\overline{co}(\mathcal{P})} \int (a\phi + b)dp = u(y^*) = \int u \circ g dp$$

So for all $p \in \Delta(S)$ such that $p \in \mathcal{P}$, $(f, [\{p\}, p]) \succeq (g, [\{p\}, p])$. Yet

$$\min_{p\in\mathcal{F}_{[\mathcal{P},c]}} \int u \circ f dp \leq \int u \circ f dp^* = \int (a\phi + b)dp < \min_{p\in\overline{co}(\mathcal{P})} \int (a\phi + b)dp = u(y^*) = \min_{p\in\mathcal{F}_{[\mathcal{P},c]}} \int u \circ g dp$$

and thus $(f, [\mathcal{P}, c]) \prec (g, [\mathcal{P}, c])$ which is a violation of axiom 12.

Condition 2 was proved in theorem 1.

Turn now to condition 3. Consider $[\mathcal{P}_i, c_i], [\mathcal{P}_j, c_j] \in \mathcal{S}$ such that $S(\mathcal{P}_i) \cap S(\mathcal{P}_j) = \emptyset$ and $\alpha \in [0, 1]$. We show first that $\mathcal{F}_{\left[\mathcal{P}_M^{(\alpha,i,j)}, c_M^{(\alpha,i,j)}\right]} \supseteq \alpha\mathcal{F}_{[\mathcal{P}_i,c_i]} + (1 - \alpha)\mathcal{F}_{[\mathcal{P}_j,c_j]}$. Suppose on the contrary that there exists $p_i^* \in \mathcal{F}_{[\mathcal{P}_i,c_i]}$ and $p_j^* \in \mathcal{F}_{[\mathcal{P}_j,c_j]}$ such that $p^* = \alpha p_i^* + (1 - \alpha)p_j^* \notin \mathcal{F}_{\left[\mathcal{P}_M^{(\alpha,i,j)}, c_M^{(\alpha,i,j)}\right]}$. Since $\mathcal{F}_{\left[\mathcal{P}_M^{(\alpha,i,j)}, c_M^{(\alpha,i,j)}\right]}$ is a convex set, using a separation argument, we know there exists a function $\phi : S \rightarrow \mathbb{R}$ such that $\int \phi dp^* < \min_{p\in\mathcal{F}_{\left[\mathcal{P}_M^{(\alpha,i,j)}, c_M^{(\alpha,i,j)}\right]}} \int \phi dp$. Since $S(\mathcal{P}_i)$ and $S(\mathcal{P}_j)$ are finite sets, there exist numbers a, b with $a > 0$, such that $\forall s \in S(\mathcal{P}_i) \cup S(\mathcal{P}_j)$, $(a\phi(s) + b) \in u(Y)$[1]. Then, for all $s \in S(\mathcal{P}_i) \cup S(\mathcal{P}_j)$ there exists $y(s) \in Y$ such that $u(y(s)) = a\phi(s) + b$. Define f by $f(s) = y(s)$ for all $s \in S(\mathcal{P}_i) \cup S(\mathcal{P}_j)$, $f(s) = \delta_x$ for all $s \in S\backslash (S(\mathcal{P}_i) \cup S(\mathcal{P}_j))$, where $x \in X$. Since for all $p \in \mathcal{F}_{\left[\mathcal{P}_M^{(\alpha,i,j)}, c_M^{(\alpha,i,j)}\right]}$, $p(S(\mathcal{P}_M^{(\alpha,i,j)})) = p(S(\mathcal{P}_i) \cup S(\mathcal{P}_j)) = 1$, then

$$\begin{aligned}
\min_{p\in\mathcal{F}_{[\mathcal{P}_M^{(\alpha,i,j)}, c_M^{(\alpha,i,j)}]}} \int u \circ f dp &= \min_{p\in\mathcal{F}_{[\mathcal{P}_M^{(\alpha,i,j)}, c_M^{(\alpha,i,j)}]}} \int (a\phi + b)dp \\
&> \int (a\phi + b)dp^* = \alpha \int u \circ f dp_i^* + (1 - \alpha) \int u \circ f dp_j^*
\end{aligned}$$

Since $\int u \circ f dp_i^* \in u(Y)$ and $\int u \circ f dp_j^* \in u(Y)$ there exists $y_i, y_j \in Y$ such that $u(y_i) = \int u \circ f dp_i^*$ and $u(y_j) = \int u \circ f dp_j^*$. Define g by $g(s) = y_i$ for all $s \in S(\mathcal{P}_i)$, $g(s) = y_j$ for all $s \in S\backslash S(\mathcal{P}_i)$. Since for all $p \in \mathcal{F}_{[\mathcal{P}_i,c_i]}$, $p(S(\mathcal{P}_i)) = 1$ we have that

$$\min_{p\in\mathcal{F}_{[\mathcal{P}_i,c_i]}} \int u \circ g dp = u(y_i) = \int u \circ f dp_i^* \geq \min_{p\in\mathcal{F}_{[\mathcal{P}_i,c_i]}} \int u \circ f dp$$

[1]Completeness and continuity imply that $u(Y)$ is convex.

and

$$\min_{p\in\mathcal{F}_{[\mathcal{P}_j,c_j]}}\int u\circ gdp = u(y_j) = \int u\circ fdp_j^* \geq \min_{p\in\mathcal{F}_{[\mathcal{P}_j,c_j]}}\int u\circ fdp$$

thus $(g,[\mathcal{P}_i,c_i]) \succeq (f,[\mathcal{P}_i,c_i])$ and $(g,[\mathcal{P}_j,c_j]) \succeq (f,[\mathcal{P}_j,c_j])$.

On the other hand, since

$$\begin{aligned}\min_{p\in\mathcal{F}_{\left[\mathcal{P}_M^{(\alpha,i,j)},c_M^{(\alpha,i,j)}\right]}}\int u\circ fdp &> \alpha\int u\circ fdp_i^* + (1-\alpha)\int u\circ fdp_j^* \\ &= \alpha u(y_i) + (1-\alpha)u(y_j) = \int u\circ gd(\alpha c_i + (1-\alpha)c_j)\end{aligned}$$

we get that $\left(f,\left[\mathcal{P}_M^{(\alpha,i,j)},c_M^{(\alpha,i,j)}\right]\right) \succ (g,[\{\alpha c_i+(1-\alpha)c_j\},\alpha c_i+(1-\alpha)c_j])$. Remark that the situation $[\{\alpha c_i+(1-\alpha)c_j\},\alpha c_i+(1-\alpha)c_j]$ is less imprecise than $\left[\mathcal{P}_M^{(\alpha,i,j)},c_M^{(\alpha,i,j)}\right]$, since $c_M^{(\alpha,i,j)} = \alpha c_i+(1-\alpha)c_j$ and therefore, by axiom 9 we have that $(g,[\{\alpha c_i+(1-\alpha)c_j\},\alpha c_i+(1-\alpha c_j)]) \succeq \left(g,\left[\mathcal{P}_M^{(\alpha,i,j)},c_M^{(\alpha,i,j)}\right]\right)$ and hence $\left(f,\left[\mathcal{P}_M^{(\alpha,i,j)},c_M^{(\alpha,i,j)}\right]\right) \succ \left(g,\left[\mathcal{P}_M^{(\alpha,i,j)},c_M^{(\alpha,i,j)}\right]\right)$.

Using axiom 7 together with what we established above, we have that:

$$\begin{gathered}(g,[\mathcal{P}_i,c_i]) \succeq \left(f_{S(\mathcal{P}_i)}g,[\mathcal{P}_i,c_i]\right) \sim (f,[\mathcal{P}_i,c_i]) \\ \left(f_{S(\mathcal{P}_j)}g,[\mathcal{P}_j,c_j]\right) \sim (g,[\mathcal{P}_j,c_j]) \succeq (f,[\mathcal{P}_j,c_j])\end{gathered}$$

Axiom 10 (taking $\mathcal{P}_1=\mathcal{P}_2=\mathcal{P}_i$ and $\mathcal{P}_3=\mathcal{P}_j$) then implies that:

$$\left(f_{S(\mathcal{P}_i)}g,\left[\mathcal{P}_M^{(\alpha,i,j)},c_M^{(\alpha,i,j)}\right]\right) \succeq \left(f,\left[\mathcal{P}_M^{(\alpha,i,j)},c_M^{(\alpha,i,j)}\right]\right)$$

and

$$\left(g,\left[\mathcal{P}_M^{(\alpha,i,j)},c_M^{(\alpha,i,j)}\right]\right) \succeq \left(f_{S(\mathcal{P}_i)}g,\left[\mathcal{P}_M^{(\alpha,i,j)},c_M^{(\alpha,i,j)}\right]\right)$$

implying that $\left(g,\left[\mathcal{P}_M^{(\alpha,i,j)},c_M^{(\alpha,i,j)}\right]\right) \succeq \left(f,\left[\mathcal{P}_M^{(\alpha,i,j)},c_M^{(\alpha,i,j)}\right]\right)$, a contradiction.

Let us show now that $\mathcal{F}_{\left[\mathcal{P}_M^{(\alpha,i,j)},c_M^{(\alpha,i,j)}\right]} \subseteq \alpha\mathcal{F}_{[\mathcal{P}_i,c_i]} + (1-\alpha)\mathcal{F}_{[\mathcal{P}_j,c_j]}$. Suppose on the contrary that there exists $p^*\in\mathcal{F}_{\left[\mathcal{P}_M^{(\alpha,i,j)},c_M^{(\alpha,i,j)}\right]}$ such that $p^*\notin\alpha\mathcal{F}_{[\mathcal{P}_i,c_i]} + (1-\alpha)\mathcal{F}_{[\mathcal{P}_j,c_j]}$. Suppose first that $p^*(S(\mathcal{P}_i))\neq\alpha$, for instance, $p^*(S(\mathcal{P}_i))>\alpha$. There exists $y_1,y_2,y_3\in Y$ such that $\alpha u(y_1)+(1-\alpha)u(y_2)=u(y_3)$[2]. Define f by $f(s)=y_1$ for all $s\in S(\mathcal{P}_i)$, $f(s)=y_2$ for all $s\in S\backslash S(\mathcal{P}_i)$ and g by $g(s)=y_3$ for all $s\in S$. Then

$$\min_{p\in\mathcal{F}_{\left[\mathcal{P}_M^{(\alpha,i,j)},c_M^{(\alpha,i,j)}\right]}}\int u\circ fdp \leq \int u\circ fdp^* < \alpha u(y_1)+(1-\alpha)u(y_2) = u(y_3) = \min_{p\in\mathcal{F}_{\left[\mathcal{P}_M^{(\alpha,i,j)},c_M^{(\alpha,i,j)}\right]}}\int u\circ gdp$$

and thus $\left(f,\left[\mathcal{P}_M^{(\alpha,i,j)},c_M^{(\alpha,i,j)}\right]\right) \prec \left(g,\left[\mathcal{P}_M^{(\alpha,i,j)},c_M^{(\alpha,i,j)}\right]\right)$. Without loss of generality, let $S(\mathcal{P}_i)=\{1,...,n_1\}$ and $S(\mathcal{P}_j)=\{n_1+1,...,n_1+n_2\}$. Consider the onto mapping $\psi: S\to S$ such that $\psi(s)=1$ for all $s\in\{1,...,n_1\}$, $\psi(s)=2$ for all $s\in\{n_1+1,...,n_1+n_2\}$, $\psi(s)=s+2-$

[2]Existence can be proved by using axioms 1,3,5 .

(n_1+n_2) for all $s > n_1+n_2$. Since f is ψ-measurable, $\left(f,\left[\mathcal{P}_M^{(\alpha,i,j)},c_M^{(\alpha,i,j)}\right]\right)$ is equivalent to $\left(f^\psi,\left(\left[(\mathcal{P}_M^{(\alpha,i,j)})^\psi,(c_M^{(\alpha,i,j)})^\psi\right]\right)\right)$. Remark that $(\mathcal{P}_M^{(\alpha,i,j)})^\psi = \{\overline{p}\}$ where $\overline{p} \in \Delta(S)$ is such that $\overline{p}(1)=\alpha$ and $\overline{p}(2)=1-\alpha$. By axiom 7, we have that $\left(f,\left[\mathcal{P}_M^{(\alpha,i,j)},c_M^{(\alpha,i,j)}\right]\right) \sim \left(f^\psi,[\{\overline{p}\},\overline{p}]\right)$. Note also, that $[\{\overline{p}\},\overline{p}]$ is the (α,φ)-replication of $[\{\delta_1\},\delta_1]$ with φ the bijection such that $\varphi(1)=2$, $\varphi(2)=1$, $\varphi(s)=s$ for all $s>2$. Define h by $h(s)=\alpha y_1+(1-\alpha)y_2$ for all $s\in S$. By axiom 8, $\left(f^\psi,[\{\overline{p}\},\overline{p}]\right) \sim \left(\left(f^\psi\right)_{\{1,2\}}h,[\{\overline{p}\},\overline{p}]\right)$. By axiom 11, $\left(\left(f^\psi\right)_{\{1,2\}}h,[\{\overline{p}\},\overline{p}]\right) \sim (h,[\{\delta_1\},\delta_1])$. Since $\min_{p\in\mathcal{F}_{[\{\delta_1\},\delta_1]}}\int u\circ h dp = \alpha u(y_1)+(1-\alpha)u(y_2) = u(y_3)$, we have that $(h,[\{\delta_1\},\delta_1]) \sim \left(g,\left[\mathcal{P}_M^{(\alpha,i,j)},c_M^{(\alpha,i,j)}\right]\right)$. Thus we have that $\left(f,\left[\mathcal{P}_M^{(\alpha,i,j)},c_M^{(\alpha,i,j)}\right]\right) \sim \left(g,\left[\mathcal{P}_M^{(\alpha,i,j)},c_M^{(\alpha,i,j)}\right]\right)$ which is a contradiction. Therefore, $p^*\left(S(\mathcal{P}_i)\right) = (1-p^*\left(S(\mathcal{P}_j)\right) = \alpha$. Define $p_k^*\in\Delta(S)$ by $p_k^*(s) = \frac{p^*(s)}{p^*(S(\mathcal{P}_k))}$ if $s\in S(\mathcal{P}_k)$, $p_k^*(s)=0$ otherwise. At least, there exists $k\in\{i,j\}$ such that $p_k^*\notin\mathcal{F}_{[\mathcal{P}_k,c_k]}$ (otherwise, we would have $p^*\in\alpha\mathcal{F}_{[\mathcal{P}_i,c_i]}+(1-\alpha)\mathcal{F}_{[\mathcal{P}_j,c_j]}$). Suppose for instance that $p_i^*\notin\mathcal{F}_{[\mathcal{P}_i,c_i]}$. Since $\mathcal{F}_{[\mathcal{P}_i,c_i]}$ is a convex set, using a separation argument, we know there exists a function $\phi: S\to\mathbb{R}$ such that $\int\phi dp_i^* < \min_{p\in\mathcal{F}_{[\mathcal{P}_i,c_i]}}\int\phi dp$. Since $S(\mathcal{P}_i)$ is a finite set, there exist numbers a,b with $a>0$, such that $\forall s\in S(\mathcal{P}_i)$, $(a\phi(s)+b)\in u(Y)$. Then, for all $s\in S(\mathcal{P}_i)$ there exists $y(s)\in Y$ such that $u(y(s)) = a\phi(s)+b$. There also exists $y^*\in Y$ such that $u(y^*) = \min_{p\in\mathcal{F}_{[\mathcal{P}_i,c_i]}}\int(a\phi+b)\,dp$. Define f by $f(s)=y(s)$ for all $s\in S(\mathcal{P}_i)$, $f(s)=y^*$ for all $s\in S\backslash S(\mathcal{P}_i)$ and define g by $g(s)=y^*$ for all $s\in S$. Since condition 1 applies, we have that

$$\begin{aligned}\min_{p\in\mathcal{F}_{[\mathcal{P}_i,c_i]}}\int u\circ f dp &= \min_{p\in\mathcal{F}_{[\mathcal{P}_j,c_j]}}\int u\circ f dp\\ \min_{p\in\mathcal{F}_{[\mathcal{P}_i,c_i]}}\int u\circ g dp &= \min_{p\in\mathcal{F}_{[\mathcal{P}_j,c_j]}}\int u\circ g dp = u(y^*)\end{aligned}$$

Thus $(f,[\mathcal{P}_i,c_i]) \sim (g,[\mathcal{P}_i,c_i]) \sim (f,[\mathcal{P}_j,c_j]) \sim (g,[\mathcal{P}_j,c_j])$. By axiom 7, $(f,[\mathcal{P}_i,c_i]) \sim \left(f_{S(\mathcal{P}_i)}g,[\mathcal{P}_i,c_i]\right)$. By axiom 10,

$$\left(f_{S(\mathcal{P}_i)}g,\left[\mathcal{P}_M^{(\alpha,i,j)},c_M^{(\alpha,i,j)}\right]\right) \sim \left(f,\left[\mathcal{P}_M^{(\alpha,i,j)},c_M^{(\alpha,i,j)}\right]\right)$$

and

$$\left(f_{S(\mathcal{P}_i)}g,\left[\mathcal{P}_M^{(\alpha,i,j)},c_M^{(\alpha,i,j)}\right]\right) \sim \left(g,\left[\mathcal{P}_M^{(\alpha,i,j)},c_M^{(\alpha,i,j)}\right]\right)$$

establishing that $\left(f,\left[\mathcal{P}_M^{(\alpha,i,j)},c_M^{(\alpha,i,j)}\right]\right) \sim \left(g,\left[\mathcal{P}_M^{(\alpha,i,j)},c_M^{(\alpha,i,j)}\right]\right)$. Since g is a constant act, we have $\min_{p\in\mathcal{F}_{[\mathcal{P}_M^{(\alpha,i,j)},c_M^{(\alpha,i,j)}]}}\int u\circ g dp = u(y^*)$. Yet,

$$\begin{aligned}\min_{p\in\mathcal{F}_{[\mathcal{P}_M^{(\alpha,i,j)},c_M^{(\alpha,i,j)}]}}\int u\circ f dp &\leq \int u\circ f dp^* = \alpha\int u\circ f dp_i^* + (1-\alpha)\int u\circ f dp_j^*\\ &= \alpha\int(a\phi+b)\,dp_i^* + (1-\alpha)u(y^*)\\ &< \alpha\min_{p\in\mathcal{F}_{[\mathcal{P}_i,c_i]}}\int(a\phi+b)\,dp + (1-\alpha)u(y^*) = u(y^*)\end{aligned}$$

which is a contradiction to the fact that $\left(f,\left[\mathcal{P}_M^{(\alpha,i,j)},c_M^{(\alpha,i,j)}\right]\right)$ and $\left(g,\left[\mathcal{P}_M^{(\alpha,i,j)},c_M^{(\alpha,i,j)}\right]\right)$ are indifferent.

To prove condition 4, consider $[\mathcal{P}_i, c_i], [\mathcal{P}_j, c_j] \in \mathcal{S}$ where $[\mathcal{P}_i, c_i]$ is a center preserving increase in imprecision of $[\mathcal{P}_j, c_j]$ (and hence $c_i = c_j$). Suppose on the contrary that $\mathcal{F}_{[\mathcal{P}_i,c_i]} \not\supseteq \mathcal{F}_{[\mathcal{P}_j,c_j]}$ and thus, that there exists $p^* \in \mathcal{F}_{[\mathcal{P}_j,c_j]}$ such that $p^* \notin \mathcal{F}_{[\mathcal{P}_i,c_i]}$. Since $\mathcal{F}_{[\mathcal{P}_i,c_i]}$ is a convex set, using a separation argument, we know there exists a function $\phi : S \to \mathbb{R}$ such that $\int \phi dp^* < \min_{p \in \mathcal{F}_{[\mathcal{P}_i,c_i]}} \int \phi dp$. Since $S(\mathcal{P}_i)$ is a finite set, there exist numbers a, b with $a > 0$, such that $\forall s \in S(\mathcal{P}_i)$, $(a\phi(s) + b) \in u(Y)$. Then, for all $s \in S(\mathcal{P}_i)$ there exists $y(s) \in Y$ such that $u(y(s)) = a\phi(s) + b$. Define f by $f(s) = y(s)$ for all $s \in S(\mathcal{P}_\rangle)$, $f(s) = \delta_x$ for all $s \in S \backslash S(\mathcal{P}_i)$, where $x \in X$. Note that since $[\mathcal{P}_i, c_i]$ is a center preserving increase in imprecision of $[\mathcal{P}_j, c_j]$, $S(\mathcal{P}_j) \subseteq S(\mathcal{P}_i)$ and thus we have

$$\begin{aligned}\min_{p \in \mathcal{F}_{[\mathcal{P}_j,c_j]}} \int u \circ f dp &\leq \int u \circ f dp^* = \int (a\phi + b)\, dp^* \\ &< \min_{p \in \mathcal{F}_{[\mathcal{P}_i,c_i]}} \int (a\phi + b)\, dp = \min_{p \in \mathcal{F}_{[\mathcal{P}_i,c_i]}} \int u \circ f dp\end{aligned}$$

which implies that $(f, [\mathcal{P}_i, c_i]) \succ (f, [\mathcal{P}_j, c_j])$ which is a violation of axiom 9.

To prove condition 5, consider $[\mathcal{P}, c] \in \mathcal{S}$, and $\left[\mathcal{P}_R^{(\alpha,\varphi)}, c_R^{(\alpha,\varphi)}\right]$ a replication ($\alpha \in]0,1[$)[3]. We know by lemma 3 that $\left[\mathcal{P}_R^{(\alpha,\varphi)}, c_R^{(\alpha,\varphi)}\right]$ is less imprecise than $\left[\mathcal{P}_M^{(\alpha,\varphi)}, c_M^{(\alpha,\varphi)}\right]$ and thus, since condition 4 holds, $\mathcal{F}_{\left[\mathcal{P}_R^{(\alpha,\varphi)}, c_R^{(\alpha,\varphi)}\right]} \subseteq \mathcal{F}_{\left[\mathcal{P}_M^{(\alpha,\varphi)}, c_M^{(\alpha,\varphi)}\right]}$ and by condition 1 $\mathcal{F}_{\left[\mathcal{P}_R^{(\alpha,\varphi)}, c_R^{(\alpha,\varphi)}\right]} = \alpha\mathcal{F}_{[\mathcal{P},c]} + (1-\alpha)\mathcal{F}_{[\mathcal{P}^\varphi,c^\varphi]}$. Thus $\mathcal{F}_{\left[\mathcal{P}_R^{(\alpha,\varphi)}, c_R^{(\alpha,\varphi)}\right]} \subseteq \alpha\mathcal{F}_{[\mathcal{P},c]} + (1-\alpha)\mathcal{F}_{[\mathcal{P}^\varphi,c^\varphi]}$.

Suppose first that $\mathcal{F}_{\left[\mathcal{P}_R^{(\alpha,\varphi)}, c_R^{(\alpha,\varphi)}\right]} \nsubseteq \{\alpha p + (1-\alpha)p^\varphi | p \in \mathcal{F}_{[\mathcal{P},c]}\} = \left(\mathcal{F}_{[\mathcal{P},c]}\right)^\varphi$ and that there exists $p^* \in \mathcal{F}_{\left[\mathcal{P}_R^{(\alpha,\varphi)}, c_R^{(\alpha,\varphi)}\right]}$ such that $p^* \notin \left(\mathcal{F}_{[\mathcal{P},c]}\right)^{(\alpha,\varphi)}$. Then, since $p^* \in \alpha\mathcal{F}_{[\mathcal{P},c]} + (1-\alpha)\mathcal{F}_{[\mathcal{P}^\varphi,c^\varphi]}$ and since condition 2 holds, there exist $p_1^*, p_2^* \in \mathcal{F}_{[\mathcal{P},c]}$ such that, $p^* = \alpha p_1^* + (1-\alpha)p_2^{*\varphi}$. Note that $p_1^* \neq p_2^*$ since otherwise we would have $p^* \in \left(\mathcal{F}_{[\mathcal{P},c]}\right)^{(\alpha,\varphi)}$.

Thus, there exists $E_1, E_2 \subset S(\mathcal{P})$ such that $E_1 \cap E_2 = \emptyset$, $E_1 \cup E_2 = S(\mathcal{P})$, $p_1^*(E_1) > p_2^*(E_1)$ (and thus $p_1^*(E_2) = (1 - p_1^*(E_1)) < p_2^*(E_2) = (1 - p_1^*(E_2)))$. There also exist $\overline{x}$ and $\underline{x}$ $u(\delta_{\overline{x}}) > u(\delta_{\underline{x}})$.

In the case $\alpha \geq \frac{1}{2}$, define f by $f(s) = \left(\frac{2\alpha-1}{\alpha}\right)\delta_{\overline{x}} + \left(\frac{1-\alpha}{\alpha}\right)\delta_{\underline{x}}$ for all $s \in E_1$, $f(s) = \delta_{\underline{x}}$ for all $s \in E_2$, $f(s) = \alpha\delta_{\overline{x}} + (1-\alpha)\delta_{\underline{x}}$ for all $s \in S \backslash S(\mathcal{P})$ and define g by $g(s) = \delta_{\overline{x}}$ for all $s \in E_1$, $g(s) = \delta_{\underline{x}}$ for all $s \in E_2$, $g(s) = \alpha\delta_{\overline{x}} + (1-\alpha)\delta_{\underline{x}}$ for all $s \in S \backslash \{s_1, s_2\}$.

In the case $\alpha \leq \frac{1}{2}$, define f by $f(s) = \delta_{\underline{x}}$ for all $s \in E_1$, $f(s) = \delta_{\overline{x}}$ for all $s \in E_2$, $f(s) = \alpha\delta_{\overline{x}} + (1-\alpha)\delta_{\underline{x}}$ for all $s \in S \backslash S(\mathcal{P})$ and define g by $g(s) = \left(\frac{\alpha}{1-\alpha}\right)\delta_{\overline{x}} + \left(\frac{1-2\alpha}{1-\alpha}\right)\delta_{\underline{x}}$ for all $s \in E_1$, $g(s) = \delta_{\underline{x}}$ for all $s \in E_2$, $g(s) = \alpha\delta_{\overline{x}} + (1-\alpha)\delta_{\underline{x}}$ for all $s \in S \backslash S(\mathcal{P})$. In both cases, we can check that $(\alpha f + (1-\alpha)g)(s) = \alpha\delta_{\overline{x}} + (1-\alpha)\delta_{\underline{x}}$ for all $s \in S(\mathcal{P})$. And thus

$$\min_{p \in \mathcal{F}_{[\mathcal{P},c]}} \int u \circ (\alpha f + (1-\alpha)g)\, dp = u\left(\alpha\delta_{\overline{x}} + (1-\alpha)\delta_{\underline{x}}\right) = \alpha u(\delta_{\overline{x}}) + (1-\alpha)u(\delta_{\underline{x}})$$

[3]For $\alpha = 0$ we have trivially $\left[\mathcal{P}_R^{(0,\varphi)}, c_R^{(0,\varphi)}\right] = [\mathcal{P}, c]$ and for $\alpha = 1$, condition 5 can be deduced from condition 2.

Consider now $f_{S(\mathcal{P})}g^{\varphi}$. In the case $\alpha \geq \frac{1}{2}$, we have that

$$\begin{aligned}
\min_{p\in\mathcal{F}_{[\mathcal{P}_R^{(\alpha,\varphi)},c_R^{(\alpha,\varphi)}]}} \int u\circ f_{S(\mathcal{P})}g^{\varphi}dp &\leq \int u\circ f_{S(\mathcal{P})}g^{\varphi}dp^* = \alpha\int u\circ f dp_1^* + (1-\alpha)\int u\circ g dp_2^* \\
&= \alpha\left[p_1^*(E_1)u(\left(\frac{2\alpha-1}{\alpha}\right)\delta_{\overline{x}} + \left(\frac{1-\alpha}{\alpha}\right)\delta_{\underline{x}}) + p_1^*(E_2)u(\delta_{\overline{x}})\right] \\
&+ (1-\alpha)\left[p_2^*(E_1)u(\delta_{\overline{x}}) + p_2^*(E_2)u(\delta_{\underline{x}})\right] \\
&= \alpha u(\delta_{\overline{x}}) + (1-\alpha)u(\delta_{\underline{x}}) \\
&+ (1-\alpha)\left(p_2^*(E_1) - p_1^*(E_1)\right)\left(u(\delta_{\overline{x}}) - u(\delta_{\underline{x}})\right) \\
&< \alpha u(\delta_{\overline{x}}) + (1-\alpha)u(\delta_{\underline{x}})
\end{aligned}$$

and thus $(\alpha f + (1-\alpha)g, [\mathcal{P}, c]) \succ \left(f_{S(\mathcal{P})}g^{\varphi}, \left[\mathcal{P}_R^{(\alpha,\varphi)}, c_R^{(\alpha,\varphi)}\right]\right)$ which is violation of axiom 11. Similarly, in the case $\alpha \leq \frac{1}{2}$, we have that

$$\begin{aligned}
\min_{p\in\mathcal{F}_{[\mathcal{P}_R^{(\alpha,\varphi)},c_R^{(\alpha,\varphi)}]}} \int u\circ f_{S(\mu)}g^{\varphi}dp &\leq \int u\circ f_{S(\mathcal{P})}g^{\varphi}dp^* = \alpha\int u\circ f dp_1^* + (1-\alpha)\int u\circ g dp_2^* \\
&= \alpha\left[p_1^*(E_1)u(\delta_{\underline{x}}) + p_1^*(E_2)u(\delta_{\overline{x}})\right] \\
&+ (1-\alpha)\left[p_2^*(E_1)u(\left(\frac{\alpha}{1-\alpha}\right)\delta_{\overline{x}} + \left(\frac{1-2\alpha}{1-\alpha}\right)\delta_{\underline{x}}) + p_2^*(E_2)u(\delta_{\underline{x}})\right] \\
&= \alpha u(\delta_{\overline{x}}) + (1-\alpha)u(\delta_{\underline{x}}) \\
&+ \alpha\left(p_2^*(E_1) - p_1^*(E_1)\right)\left(u(\delta_{\overline{x}}) - u(\delta_{\underline{x}})\right) \\
&< \alpha u(\delta_{\overline{x}}) + (1-\alpha)u(\delta_{\underline{x}})
\end{aligned}$$

and thus $(\alpha f + (1-\alpha)g, [\mathcal{P}, c]) \succ (f_{S(\mathcal{P})}g^{\varphi}, \left[\mathcal{P}_R^{(\alpha,\varphi)}, c_R^{(\alpha,\varphi)}\right])$ which is violation of axiom 11.

Suppose now that $\mathcal{F}_{\left[\mathcal{P}_R^{(\alpha,\varphi)},c_R^{(\alpha,\varphi)}\right]} \not\supseteq \left\{\alpha p + (1-\alpha)p^{\varphi} | p \in \mathcal{F}_{[\mathcal{P},c]}\right\}$ and that there exists $p^* \in \mathcal{F}_{[\mathcal{P},c]}$ such that $\alpha p^* + (1-\alpha)p^{*\varphi} \notin \mathcal{F}_{\left[\mathcal{P}_R^{(\alpha,\varphi)},c_R^{(\alpha,\varphi)}\right]}$. Since we just proved that

$$\mathcal{F}_{\left[\mathcal{P}_R^{(\alpha,\varphi)},c_R^{(\alpha,\varphi)}\right]} \subseteq \{\alpha p + (1-\alpha)p^{\varphi} | p \in \mathcal{F}_{\mu}\}$$

for all $p \in \mathcal{F}_{\left[\mathcal{P}_R^{(\alpha,\varphi)},c_R^{(\alpha,\varphi)}\right]}$, there exists $p_{\varphi^{-1}} \in \mathcal{F}_{[\mathcal{P},c]}$ such that $p = \alpha p_{\varphi^{-1}} + (1-\alpha)\left(p_{\varphi^{-1}}\right)^{\varphi}$. Consider $\mathcal{F} = \left\{p_{\varphi^{-1}} | p \in \mathcal{F}_{\left[\mathcal{P}_R^{(\alpha,\varphi)},c_R^{(\alpha,\varphi)}\right]}\right\}$. Since $\mathcal{F}_{\left[\mathcal{P}_R^{(\alpha,\varphi)},c_R^{(\alpha,\varphi)}\right]}$ is a convex set, $\mathcal{F}$ is also convex and $p^* \notin \mathcal{F}$. Hence, using a separation argument, we know there exists a function $\phi : S \to \mathbb{R}$ such that $\int \phi dp^* < \min_{p\in\mathcal{F}} \int \phi dp$. Since $S(\mathcal{P})$ is a finite set, there exist numbers a, b with $a > 0$, such that $\forall s \in S(\mathcal{P})$, $(a\phi(s) + b) \in u(Y)$. Then, for all $s \in S(\mathcal{P})$ there exists $y(s) \in Y$ such that $u(y(s)) = a\phi(s) + b$. Define f by $f(s) = y(s)$ for all $s \in S(\mathcal{P})$, $f(s) = \delta_x$ for all $s \in S\backslash S(\mathcal{P}^{\varphi})$, where $x \in X$. Observe that for all $p \in \mathcal{F}_{\left[\mathcal{P}_R^{(\alpha,\varphi)},c_R^{(\alpha,\varphi)}\right]}$,

$$\begin{aligned}
\int u \circ f_{S(\mathcal{P})} f^{\varphi} dp &= \int u \circ f_{S(\mathcal{P})} f^{\varphi} d\left(\alpha p_{\varphi^{-1}} + (1-\alpha)\left(p_{\varphi^{-1}}\right)^{\varphi}\right) \\
&= \alpha \int u \circ f dp_{\varphi^{-1}} + (1-\alpha) \int u \circ f^{\varphi} d\left(p_{\varphi^{-1}}\right)^{\varphi} \\
&= \int u \circ f dp_{\varphi^{-1}} = \int (a\phi + b) dp_{\varphi^{-1}}
\end{aligned}$$

Thus

$$\begin{aligned}
\min_{p \in \mathcal{F}_{[\mathcal{P},c]}} \int u \circ f dp &\leq \int u \circ f dp^{*} = \int (a\phi + b) dp^{*} < \min_{p \in \mathcal{F}} \int (a\phi + b) dp \\
&= \min_{p \in \mathcal{F}} \int u \circ f dp = \min_{p \in \mathcal{F}_{\left[\mathcal{P}_R^{(\alpha,\varphi)}, c_R^{(\alpha,\varphi)}\right]}} \int u \circ f_{S(\mathcal{P})} f^{\varphi} dp
\end{aligned}$$

which shows that $\left(f_{S(\mu)} f^{\varphi}, \left[\mathcal{P}_R^{(\alpha,\varphi)}, c_R^{(\alpha,\varphi)}\right]\right) \succ (f, [\mathcal{P}, c])$, a violation of axiom 11.

($\Leftarrow$)The axioms to check are axioms 9, 10, 11, and 12 since the others hold by theorem 1.

By condition 4, if $[\mathcal{P}_i, c_i]$ is acenter preserving increase in imprecision of $[\mathcal{P}_j, c_j]$ then $\mathcal{F}_{[\mathcal{P}_i, c_i]} \supseteq \mathcal{F}_{[\mathcal{P}_j, c_j]}$ implies that for all $f \in \mathcal{A}$,

$$\min_{p \in \mathcal{F}_{[\mathcal{P}_i, c_i]}} \int u \circ f dp \leq \min_{p \in \mathcal{F}_{[\mathcal{P}_j, c_j]}} \int u \circ f dp$$

and thus $(f, [\mathcal{P}_j, c_j]) \succeq (f, [\mathcal{P}_i, c_i])$ which proves that axiom 9 is satisfied.

To prove that axiom 10 is satisfied, consider $[\mathcal{P}_i, c_i] \in \mathcal{S}$, $i = 1, 2, 3$ such that $(S(\mathcal{P}_1) \cup S(\mathcal{P}_2)) \cap S(\mathcal{P}_3) = \emptyset$, $\alpha \in [0, 1]$, and $f, g \in \mathcal{A}$ such that $f(s) = g(s)$ for all $s \in S(\mathcal{P}_3)$. Assume $(f, [\mathcal{P}_1, c_1]) \succeq (g, [\mathcal{P}_2, c_2])$. By condition 3, $\mathcal{F}_{\left[\mathcal{P}_M^{(\alpha,1,3)}, c_M^{(\alpha,1,3)}\right]} = \alpha \mathcal{F}_{[\mathcal{P}_1, c_1]} + (1-\alpha) \mathcal{F}_{[\mathcal{P}_3, c_3]}$ we have that

$$\begin{aligned}
\min_{p \in \mathcal{F}_{\left[\mathcal{P}_M^{(\alpha,1,3)}, c_M^{(\alpha,1,3)}\right]}} \int u \circ f dp &= \min_{p \in \alpha \mathcal{F}_{[\mathcal{P}_1, c_1]} + (1-\alpha) \mathcal{F}_{[\mathcal{P}_3, c_3]}} \int u \circ f dp \\
&= \alpha \min_{p \in \mathcal{F}_{[\mathcal{P}_1, c_1]}} \int u \circ f dp + (1-\alpha) \min_{p \in \mathcal{F}_{[\mathcal{P}_3, c_3]}} \int u \circ f dp \\
&\geq \alpha \min_{p \in \mathcal{F}_{[\mathcal{P}_2, c_2]}} \int u \circ g dp + (1-\alpha) \min_{p \in \mathcal{F}_{[\mathcal{P}_3, c_3]}} \int u \circ f dp
\end{aligned}$$

Since we have also

$$\min_{p \in \mathcal{F}_{\left[\mathcal{P}_M^{(\alpha,2,3)}, c_M^{(\alpha,2,3)}\right]}} \int u \circ g dp = \alpha \min_{p \in \mathcal{F}_{[\mathcal{P}_2, c_2]}} \int u \circ g dp + (1-\alpha) \min_{p \in \mathcal{F}_{[\mathcal{P}_3, c_3]}} \int u \circ f dp$$

thus $\left(f, \left[\mathcal{P}_M^{(\alpha,1,3)}, c_M^{(\alpha,1,3)}\right]\right) \succeq \left(g, \left[\mathcal{P}_M^{(\alpha,2,3)}, c_M^{(\alpha,2,3)}\right]\right)$ which proves that axiom 10 is satisfied.

To prove that axiom 11 is satisfied, consider $[\mathcal{P},c]\in\mathcal{S}$, the replication $\left[\mathcal{P}_R^{(\alpha,\varphi)},c_R^{(\alpha,\varphi)}\right]$, $f,g\in\mathcal{A}$ such that $f(s)=g(s)$ for all $s\in S\backslash S(\mathcal{P})$. Since condition 5 holds,

$$\min_{p\in\mathcal{F}_{[\mathcal{P}_R^{(\alpha,\varphi)},c_R^{(\alpha,\varphi)}]}}\int u\circ f_{S(\mathcal{P})}g^{\varphi}dp=\min_{p\in\mathcal{F}_{[\mathcal{P},c]}}\int u\circ f_{S(\mathcal{P})}g^{\varphi}d(\alpha p+(1-\alpha)p^{\varphi})$$

For all $p\in\mathcal{F}_{[\mathcal{P},c]}$,

$$\begin{aligned}\int u\circ f_{S(\mathcal{P})}g^{\varphi}d(\alpha p+(1-\alpha)p^{\varphi}) &= \alpha\int u\circ f_{S(\mathcal{P})}g^{\varphi}dp+(1-\alpha)\int u\circ f_{S(\mathcal{P})}g^{\varphi}dp^{\varphi}\\ &= \alpha\int u\circ fdp+(1-\alpha)\int u\circ gdp\\ &= \int u\circ(\alpha f+(1-\alpha)g)dp\end{aligned}$$

and thus

$$\min_{p\in\mathcal{F}_{[\mathcal{P},c]}}\int u\circ f_{S(\mathcal{P})}g^{\varphi}d(\alpha p+(1-\alpha)p^{\varphi})=\min_{p\in\mathcal{F}_{[\mathcal{P},c]}}\int u\circ(\alpha f+(1-\alpha)g)dp$$

which shows that $(\alpha f+(1-\alpha)g,[\mathcal{P},c])\sim\left(f_{S(\mu)}g^{\varphi},[\mathcal{P}_R^{(\alpha,\varphi)},c_R^{(\alpha,\varphi)}]\right)$.

To prove axiom 12, consider $f,g\in\mathcal{A}$ and $[\mathcal{P},c]\in\mathcal{S}$ such that $(f,[\{p\},p])\succeq(g,[\{p\},p])$ for all $p\in\mathcal{P}$. Remark that for all $p\in\overline{co}(\mathcal{P})$ there exists $p_1,p_2\in\Delta(S)$ and sequences $(p_1^n)_{n=1,..,\infty},(p_2^n)_{n=1,..,\infty}\in\Delta(S)$ such that $p_i^n\in\mathcal{P}$ for $i=1,2$ and $n\in\mathbb{N}$ and $\lim_{n\to\infty}p_i^n=p_i$, and $\alpha\in[0,1]$ such that $p=\alpha p_1+(1-\alpha)p_2$. Thus $\forall n\in\mathbb{N}$,

$$\begin{aligned}\int u\circ fd\left(\alpha p_1^n+(1-\alpha)p_2^n\right) &= \alpha\int u\circ fdp_1^n+(1-\alpha)\int u\circ fdp_2^n\\ &\geq \alpha\int u\circ gdp_1^n+(1-\alpha)\int u\circ gdp_2^n\\ &= \int u\circ gdd\left(\alpha p_1^n+(1-\alpha)p_2^n\right)\end{aligned}$$

and then when $n\to\infty$,

$$\int u\circ fdp\geq\int u\circ gdp$$

Since by condition 1, $\mathcal{F}_{[\mathcal{P},c]}\subseteq\overline{co}(\mathcal{P})$, for all $p\in\mathcal{F}_{[\mathcal{P},c]}$, $\int u\circ fdp\geq\int u\circ gdp$ and thus $\min_{p\in\mathcal{F}_{[\mathcal{P},c]}}\int u\circ fdp\geq\min_{p\in\mathcal{F}_{[\mathcal{P},c]}}\int u\circ gdp$ which implies that $(f,[\mathcal{P},c])\succeq(g,[\mathcal{P},c])$. ∎

Proof. [Lemma7] That axiom 12 is satisfied is straightforward.

Consider $f\in\mathcal{A}$ and $[\mathcal{P}_1,c_1],[\mathcal{P}_2,c_2]\in\mathcal{S}$ such that $[\mathcal{P}_1,c_1]$ is a center preserving increase in imprecision of $[\mathcal{P}_2,c_2]$. Since $c_1=c_2$, we have $(f,[\{c_1\},c_1])\sim(f,[\{c_2\},c_2])$. Since $\overline{co}(\mathcal{P}_1)\supseteq\overline{co}(\mathcal{P}_2)$, for all $p\in\overline{co}(\mathcal{P}_2)$, $p\in\overline{co}(\mathcal{P}_1)$ and trivially $(f,[\{p\},p])\succeq(f,[\{p\},p])$. Thus dominance implies $(f,[\mathcal{P}_2,c_2])\succeq(f,[\mathcal{P}_1,c_1])$. ∎

Proof. **[Theorem 3]** ($\Rightarrow$)By lemma 7, axioms 9 and 12 are satisfied. By theorem 2, we know there exist an unique (up to a positive linear transformation) affine function $u : Y \to \mathbb{R}$, and for all $[\mathcal{P}_1, c_1], [\mathcal{P}_1, c_1] \in \mathcal{S}$, there exist unique, non-empty, closed and convex set $\mathcal{F}_{[\mathcal{P}_1,c_1]}$ and $\mathcal{F}_{[\mathcal{P}_2,c_2]}$ of probability measures on 2^S, satisfying conditions 1 to 5 of theorem 2 and such that for all $f, g \in \mathcal{A}$, $(f, [\mathcal{P}_1, c_1]) \succeq (g, [\mathcal{P}_2, c_2])$ iff:

$$\min_{p\in\mathcal{F}_{[\mathcal{P}_1,c_1]}} \int u \circ f dp \geq \min_{p\in\mathcal{F}_{[\mathcal{P}_2,c_2]}} \int u \circ g dp.$$

Let us note that for all $[\mathcal{P}, c] \in \mathcal{S}$, for all $f, g \in \mathcal{A}$, such that

$$\underset{p\in\overline{co}(\mathcal{P})}{min} \int u \circ f dp \geq \underset{p\in\overline{co}(Supp(\mathcal{P}))}{min} \int u \circ g dp$$

and

$$\int u \circ f dc \geq \int u \circ g dc$$

it is the case that $(f, [\mathcal{P}, c]) \succeq (g, [\mathcal{P}, c])$. Indeed, since $\overline{co}(\mathcal{P})$ is closed, there exists $p \in \overline{co}(\mathcal{P})$ such that $\underset{p\in\overline{co}(\mathcal{P})}{min} \int u \circ g dp = \int u \circ g dp$. Then for all $q \in \overline{co}(\mathcal{P})$, we have that $(f, [\{q\}, q]) \succeq (g, [\{p\}, p])$ and thus $(f, [\mathcal{P}, c]) \succeq (g, [\mathcal{P}, c])$ is implied by axiom 13.

Consider $[\mathcal{P}, c] \in \mathcal{S}$, let us show that there exists $\alpha_{[\mathcal{P},c]}$ such that $\mathcal{F}_{[\mathcal{P},c]} = \overline{co}(\mathcal{P}^{\alpha})$. By condition 1 of theorem 2, we have that $\mathcal{F}_{[\mathcal{P},c]} \subseteq \overline{co}(\mathcal{P})$. On the other hand, since $[\{c\}, c]$ a center preserving decrease in imprecision of $[\mathcal{P}, c]$, by condition 4 of theorem 2, we have that$\mathcal{F}_{[\{c\},c]} \subseteq \mathcal{F}_{[\mathcal{P},c]}$ and hence $c \in \mathcal{F}_{[\mathcal{P},c]}$.

Remark also that $Supp(c) = S(\mathcal{P}^0)$ and that for all $\beta, \gamma \in [0, 1]$ such that $\beta \geq \gamma$, we have that $\overline{co}(\mathcal{P}^{\beta}) \supseteq \overline{co}(\mathcal{P}^{\gamma}))$. Thus consider the sets

$$\left\{\beta | \beta \in [0, 1] \text{ and } \overline{co}(\mathcal{P}^{\beta}) \subseteq \mathcal{F}_{[\mathcal{P},c]}\right\}$$

and

$$\left\{\beta | \beta \in [0, 1] \text{ and } \overline{co}(\mathcal{P}^{\beta}) \supseteq \mathcal{F}_{[\mathcal{P},c]}\right\}$$

Let

$$\underline{\beta} = Sup\left\{\beta | \beta \in [0, 1] \text{ and } \overline{co}(\mathcal{P}^{\beta}) \subseteq \mathcal{F}_{[\mathcal{P},c]}\right\}$$

and

$$\overline{\beta} = Inf\left\{\beta | \beta \in [0, 1] \text{ and } \overline{co}(\mathcal{P}^{\beta}) \supseteq \mathcal{F}_{[\mathcal{P},c]}\right\}$$

Since for all $\beta \in [0, 1]$, the set $\overline{co}(\mathcal{P}^{\beta})$ and $\mathcal{F}_{[\mathcal{P},c]}$ are closed set, we have that

$$\overline{co}(\mathcal{P}^{\underline{\beta}}) \subseteq \mathcal{F}_{[\mathcal{P},c]} \subseteq \overline{co}(\mathcal{P}^{\overline{\beta}})$$

If $\underline{\beta} = \overline{\beta}$, just take $\alpha_{[\mathcal{P},c]} = \underline{\beta} = \overline{\beta}$ and we have $\mathcal{F}_{\mu} = \overline{co}(\mathcal{P}^{\alpha})$.

Suppose $\underline{\beta} \neq \overline{\beta}$ and thus $\underline{\beta} < \overline{\beta}$. Consider $\beta \in \left]\underline{\beta}, \overline{\beta}\right[$. Then there exist p_1, p_2 in $\overline{co}(\mathcal{P}) \backslash Int\left(\overline{co}(\mathcal{P})\right)$ such that $(\beta p_1 + (1-\beta)c) \in \mathcal{F}_{[\mathcal{P},c]}$ and $(\beta p_2 + (1-\beta)c) \notin \mathcal{F}_{[\mathcal{P},c]}$[4]. Since $\mathcal{F}_{[\mathcal{P},c]}$ is a convex set, using a separation argument, we know there exists a function $\phi_2 : S \to \mathbb{R}$ such that $\int \phi_2 d\left(\beta p_2 + (1-\beta)c\right) < \min_{p \in \mathcal{F}_{[\mathcal{P},c]}} \int \phi_2 dp$. Since $p_1 \in \overline{co}(\mathcal{P}) \backslash Int\left(\overline{co}(\mathcal{P})\right)$ and since $\overline{co}(\mathcal{P})$ is a convex set, using a separation argument, we know there exists a function $\phi_1 : S \to \mathbb{R}$ such that $\int \phi_1 dp_1 \leq \min_{p \in \overline{co}(\mathcal{P})} \int \phi_1 dp$. Since $S(\mathcal{P})$ is a finite set, there exist numbers a_1, b_1 with $a_1 > 0$, a_2, b_2 with $a_2 > 0$, such that $\forall s \in S(\mu)$, $(a_i \phi_i(s) + b_i) \in u(Y)$ for $i = 1, 2$,

$$\begin{cases} \min_{p \in \overline{co}(\mathcal{P})} \int (a_1 \phi_1 + b_1)\, dp &= \min_{p \in \overline{co}(\mathcal{P})} \int (a_2 \phi_2 + b_2)\, dp \\ \int (a_1 \phi_1 + b_1)\, dc &= \int (a_2 \phi_2 + b_2)\, dc \end{cases}$$

Then, for all $s \in S(\mathcal{P})$ there exists $y_i(s) \in Y$ such that $u(y_i(s)) = a_i \phi_i(s) + b_i$. Define f by $f(s) = y_1(s)$ for all $s \in S(\mathcal{P})$, $f(s) = \delta_x$ for all $s \in S \backslash S(\mathcal{P})$, where $x \in X$ and g by $g(s) = y_2(s)$ for all $s \in S(\mathcal{P})$, $g(s) = \delta_x$ for all $s \in S \backslash S(\mathcal{P})$. Thus we have

$$\min_{p \in \overline{co}(\mathcal{P})} \int u \circ f dp = \min_{p \in \overline{co}(\mathcal{P})} \int u \circ g dp$$

and

$$\int u \circ f dc = \int u \circ g dc$$

and thus $(f, [\mathcal{P}, c]) \sim (g, [\mathcal{P}, c])$. On the other hand

$$\begin{aligned} \min_{p \in \mathcal{F}_{[\mathcal{P},c]}} \int u \circ g dp &> \int u \circ g d\left(\beta p_2 + (1-\beta)c\right) \\ &= \beta \int u \circ g dp_2 + (1-\beta) \int u \circ g dc \\ &\geq \beta \min_{p \in \overline{co}(\mathcal{P})} \int u \circ g dp + (1-\beta) \int u \circ g dc \\ &= \beta \min_{p \in \overline{co}(\mathcal{P})} \int u \circ f dp + (1-\beta) \int u \circ f dc \\ &= \beta \int u \circ f dp_1 + (1-\beta) \int u \circ f dc \\ &= \int u \circ f d\left(\beta p_1 + (1-\beta)c\right) \\ &\geq \min_{p \in \mathcal{F}_{[\mathcal{P},c]}} \int u \circ f dp \end{aligned}$$

which yields a contradiction.

Let us show now that for all $[\mathcal{P}_1, c_1], [\mathcal{P}_2, c_2] \in \mathcal{S}$, $\alpha_{[\mathcal{P}_1,c_1]} = \alpha_{[\mathcal{P}_2,c_2]}$. Suppose on the contrary that $\alpha_{[\mathcal{P}_1,c_1]} \neq \alpha_{[\mathcal{P}_2,c_2]}$ for $[\mathcal{P}_1, c_1], [\mathcal{P}_2, c_2] \in \mathcal{S}$. Without loss of generality, consider that $S(\mathcal{P}_1) \cap S(\mathcal{P}_2) = \emptyset$[5]. Consider $\alpha \in \left]0, 1\right[$ and $\left[\mathcal{P}_M^{(\alpha,1,2)}, c_M^{(\alpha,1,2)}\right]$. Since condition 3 of theorem 2 holds, we

[4]Such a couple (p_1, p_2) exists since otherwise, we would have either $\overline{co}(\mathcal{P}^\beta)) \subseteq \mathcal{F}_{[\mathcal{P},c]}$ or $\overline{co}(\mathcal{P}^\beta)) \supseteq \mathcal{F}_{[\mathcal{P},c]}$

[5]We can always find a bijection φ such that $S(\mathcal{P}_1) \cap S(\mathcal{P}_2^\varphi) = \emptyset$ and it can be easily check that since condition 1 of theorem 2 holds, we have that $\alpha_{[\mathcal{P}_2,c_2]} = \alpha_{[\mathcal{P}^\varphi, c^\varphi]}$.

have that

$$\mathcal{F}_{[\mathcal{P}_M^{(\alpha,1,2)},c_M^{(\alpha,1,2)}]} = \alpha\mathcal{F}_{[\mathcal{P}_1,c_1]} + (1-\alpha)\mathcal{F}_{[\mathcal{P}_2,c_2]}$$

On the other hand,

$$\mathcal{F}_{[\mathcal{P}_M^{(\alpha,1,2)},c_M^{(\alpha,1,2)}]} = \overline{co}\left(\mathcal{P}_M^{(\alpha,1,2)}\right)^{\alpha_{[\mathcal{P}_M^{(\alpha,1,2)},c_M^{(\alpha,1,2)}]}}$$

Necessarily, either $\alpha_{[\mathcal{P}_M^{(\alpha,1,2)},c_M^{(\alpha,1,2)}]} \neq \alpha_{[\mathcal{P}_1,c_1]}$ or $\alpha_{[\mathcal{P}_M^{(\alpha,1,2)},c_M^{(\alpha,1,2)}]} \neq \alpha_{[\mathcal{P}_2,c_2]}$ and we run into a contradiction. For instance, suppose $\alpha_{[\mathcal{P}_M^{(\alpha,1,2)},c_M^{(\alpha,1,2)}]} > \alpha_{[\mathcal{P}_1,c_1]}$ and consider $p_i \in \overline{co}(\mathcal{P}_i))\backslash Int\,(\overline{co}(\mathcal{P}_i)))$ for $i = 1,2$.

$$\begin{aligned}\alpha[\alpha_{[\mathcal{P}_M^{(\alpha,1,2)},c_M^{(\alpha,1,2)}]}p_1 &+ (1-\alpha_{[\mathcal{P}_M^{(\alpha,1,2)},c_M^{(\alpha,1,2)}]})c_1)]+\\ &+ (1-\alpha)\left[\alpha_{[\mathcal{P}_M^{(\alpha,1,2)},c_M^{(\alpha,1,2)}]}p_2 + (1-\alpha_{[\mathcal{P}_M^{(\alpha,1,2)},c_M^{(\alpha,1,2)}]})c_2\right] \in \mathcal{F}_{[\mathcal{P}_M^{(\alpha,1,2)},c_M^{(\alpha,1,2)}]}\end{aligned}$$

and thus

$$\left[\alpha_{[\mathcal{P}_M^{(\alpha,1,2)},c_M^{(\alpha,1,2)}]}p_1 + (1-\alpha_{[\mathcal{P}_M^{(\alpha,1,2)},c_M^{(\alpha,1,2)}]})c_1)\right] \in \mathcal{F}_{[\mathcal{P}_1,c_1]}$$

which yields a contradiction with the fact $\alpha_{[\mathcal{P}_M^{(\alpha,1,2)},c_M^{(\alpha,1,2)}]} > \alpha_{[\mathcal{P}_1,c_1]}$. The proof can be adapted to the other cases.

Finally, it can be easily checked that for all $[\mathcal{P}_1,c_1],[\mathcal{P}_2,c_2] \in \mathcal{S}$, for all $f,g \in \mathcal{A}$,

$$\min_{p\in\overline{co}(\mathcal{P}_1^{\alpha})}\int u\circ f dp = \alpha \min_{p\in\overline{co}(\mathcal{P}_1)}\int u\circ f dp + (1-\alpha)\int u\circ f dc_1$$

($\Leftarrow$)Conversely it is easy to check that conditions 1 to 5 of theorem 2 hold and thus following the preceding proofs, axioms 1, 3 to 5, 7, 10, 11.

We just have to check that axiom 13 is satisfied. Consider $f,g \in \mathcal{A}$, $[\mathcal{P}_1,c_1],[\mathcal{P}_2,c_2] \in \mathcal{S}$, such that $(f,[\{c_1\}c_1]) \succeq (g,[\{c_2\},c_2])$ and for all $p \in \overline{co}(\mathcal{P}_1)$, there exists $q \in \overline{co}(\mathcal{P}_2)$ such that $(f,[\{p\},p]) \succeq (g,[\{q\},q])$.

Given the representation obtained, this implies first that

$$\int u\circ f dc_1 \geq \int u\circ g dc_2$$

Furthermore, $\forall p \in \overline{co}(\mathcal{P}_1)$, there exists $q \in \overline{co}(\mathcal{P}_2)$ such that $\int u\circ f dp \geq \int u\circ g dq$. Hence,

$$\min_{p\in\overline{co}(\mathcal{P}_1)}\int u\circ f dp \geq \min_{q\in\overline{co}(\mathcal{P}_2)}\int u\circ g dq$$

Thus,

$$\alpha \min_{p\in\overline{co}(\mathcal{P}_1)}\int u\circ f dp + (1-\alpha)\int u\circ f dc_1 \geq \alpha \min_{p\in\overline{co}(\mathcal{P}_2)}\int u\circ f dp + (1-\alpha)\int u\circ g dc_2$$

and thus $(f,[\mathcal{P}_1,c_1]) \succeq (g,[\mathcal{P}_2,c_2])$. ■

Proof. [Theorem 4]

Let $V_i(f,[\mathcal{P},c]) = \underset{p\in\mathcal{F}^i_{[\mathcal{P},c]}}{min} \int u_i \circ f dp$, with $i \in \{a,b\}$, defined as in Theorem 2.

$[(i) \Rightarrow (ii)]$ Let $(f,[\mathcal{P},c]) \in \mathcal{A}_B \times \mathcal{S}$. By definition, $\left(k_{Pe_a(f,[\mathcal{P},c])},[\mathcal{P},c]\right) \sim_a (f,[\mathcal{P},c])$. Since $\succeq_b$ is more uncertainty averse than $\succeq_a$, one has: $(k_{Pe_a(f,[\mathcal{P},c])},[\mathcal{P},c]) \succeq_b (f,[\mathcal{P},c])$, and, by definition, $(f,[\mathcal{P},c]) \sim_b (k_{Pe_b(f,[\mathcal{P},c])},[\mathcal{P},c])$. Therefore, $(k_{Pe_a(f,[\mathcal{P},c])},[\mathcal{P},c]) \succeq_b (k_{Pe_b(f,[\mathcal{P},c])},[\mathcal{P},c])$. Hence, $u_a(k_{Pe_a(f,[\mathcal{P},c])}) \geq u_a(k_{Pe_b(f,[\mathcal{P},c])})$, i.e., using the normalization $u_i(\delta_0) = 0$ and $u_i(\delta_M) = 1$: $\Pi_a(f,[\mathcal{P},c]) \geq \Pi_b(f,[\mathcal{P},c])$.

$[(ii) \Rightarrow (iii)]$ Let $(f,[\mathcal{P},c]) \in \mathcal{A}_B \times \mathcal{S}$. Using the normalization $u_i(\delta_0) = 0$ and $u_i(\delta_M) = 1$, we get: $V_i(f,[\mathcal{P},c]) = \Pi_i(f,[\mathcal{P},c])$, with $i \in \{a,b\}$. Assume that $\Pi_a(f,[\mathcal{P},c]) \geq \Pi_b(f,[\mathcal{P},c])$. This implies $V_a(f,[\mathcal{P},c]) \geq V_b(f,[\mathcal{P},c])$, for all $(f,[\mathcal{P},c]) \in \mathcal{A}_B \times \mathcal{S}$. Therefore, using the representation given in Theorem 2[6], $\underset{p\in\mathcal{F}^a_{[\mathcal{P},c]}}{min} \int u \circ f dp \geq \underset{p\in\mathcal{F}^b_{[\mathcal{P},c]}}{min} \int u \circ f dp$. Assume $\mathcal{F}^a_{[\mathcal{P},c]} \not\subset \mathcal{F}^b_{[\mathcal{P},c]}$. Then, there exists $p^* \in \mathcal{F}^a_{[\mathcal{P},c]}$ such that $p^* \notin \mathcal{F}^b_{[\mathcal{P},c]}$. Hence, using a separation argument, we know there exists a function $\phi : S \to \mathbb{R}$ such that $\int \phi dp^* < \underset{p\in\mathcal{F}^b_{[\mathcal{P},c]}}{min} \int \phi dp$. Since $S(\mathcal{P})$ is a finite set, there exist numbers $a > 0$ and b, such that for all $s \in S(\mathcal{P})$, $(a\phi(s) + b \in u(Y))$. Then, for all $s \in S(\mathcal{P})$, there exits $y(s) \in Y_B$ such that $u(y(s)) = a\phi(s) + b$. Define f by $f(s) = y(s)$ for all $s \in S(\mathcal{P})$, and $f(s) = \delta_0$ for all $s \in S \setminus S(\mathcal{P})$. We then obtain: $\int u \circ f dp^* < \underset{p\in\mathcal{F}^b_{[\mathcal{P},c]}}{min} \int u \circ f dp$. Therefore, $\underset{p\in\mathcal{F}^a_{[\mathcal{P},c]}}{min} \int u \circ f dp < \underset{p\in\mathcal{F}^b_{[\mathcal{P},c]}}{min} \int u \circ f dp$, i.e., $V_a(f,[\mathcal{P},c]) = \Pi_a(f,[\mathcal{P},c]) < V_b(f,[\mathcal{P},c]) = V_b(f,[\mathcal{P},c])$, which yields a contradiction.

$[(iii) \Rightarrow (i)]$ Straightforward. ■

[6]Since $f \in \mathcal{A}_B$, we can, using the normalization above, restrict our attention to $u_a = u_b = u$.

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