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Unequal Uncertainties and Uncertain Inequalities : An Axiomatic Approach*

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Abstract

A large body of literature has been devoted to inequality measurement on the one hand, and to the evaluation of uncertain prospects of income on the other hand. However, very little has been written on the measurement of inequality under uncertainty.

In this paper, we provide an axiomatic characterization of social welfare functions under uncertainty. Our most general result is that a small number of reasonable assumptions regarding welfare orderings under uncertainty rule out pure *ex ante* as well as pure *ex post* evaluations. Any social welfare function that satisfies these axioms should lie *strictly* between the *ex ante* and the *ex post* evaluations of income distributions. We also provide an axiomatic characterization of the weighted average of the minimum and the maximum of *ex post* and the *ex ante* evaluations.

key words: Inequality, uncertainty.

résumé

Si la mesure des inégalités d'une part, et l'évaluation des revenus soumis à des aléas d'autre part, ont suscité un grand nombre de travaux, la mesure des inégalités en présence d'incertitude, quant à elle, n'a été que peu étudiée.

Nous proposons, dans cet article, une caractérisation axiomatique des fonctions de bien-être social en présence d'incertitude. Notre résultat le plus général est le suivant: un nombre réduit d'axiomes raisonnables portant sur le bien-être en présence d'incertitude suffit à exclure les fonctions de bien-être social fondées sur des évaluations exclusivement *ex ante* ou *ex post* des distributions de revenus. Toute fonction de bien-être social satisfaisant ces axiomes doit être comprise entre les évaluations *ex ante* et *ex post* des distributions de revenus.

key words: Inequality, uncertainty.

1 Introduction

Consider a society divided into two sectors of equal size – say, sector A and sector B. Sector A corresponds to domestic services that cannot be traded at the international level while sector B corresponds to manufacturing industries that can be traded. The government decides if international trade is allowed or not. If no international trade is permitted, whatever happens, wages remain equal to \$1000 a month in both sectors. In contrast, if international trade is allowed, wages in sector B depend on an exogeneous shock on international demand, which is positive with a probability of 1/2 and negative with a probability of 1/2. If the shock is positive, wages in sector B are \$1500 a month, whereas if the shock is negative, wages are only \$600 a month. In other words, trade is assumed to increase simultaneously total income, inequality and uncertainty. The two possible policies can be represented by the following tables.

no trade	sector A	sector B
shock > 0	1000	1000
shock < 0	1000	1000

trade	sector A	sector B
shock > 0	1000	1500
shock < 0	1000	600

The government must decide whether or not to allow international trade. Clearly, the policy which should be chosen depends on the inequality and uncertainty aversions that characterize this particular society. The optimal policy, however, also depends on when individuals' welfare is evaluated, namely *before* (*ex ante*) or *after* (*ex post*) the resolution of uncertainty. For sufficiently low risk aversion, trade is certainly the best policy *ex ante*, since it increases the expected earnings in sector B without decreasing them in sector A. On the other hand, for sufficiently high inequality aversion, trade is also no doubt the worst policy *ex post,* since it decreases the lowest wages during bad periods, without increasing them during favorable periods.

More generally, when comparing uncertain income distributions, should we look at the expected income of each person, and consider that the distribution where the inequality of expected incomes is the lowest as the best one? Or should we look at the level of inequalities associated to each possible state of the world, and consider the distribution where the expected level of inequality is the lowest as the best solution?

This problem is not new and has sometimes been labeled as the "timing-effect problem"[1] : the outcome of an allocation procedure depends on whether individuals' utility levels are evaluated before or after the resolution of uncertainty. As stated by Myerson,

> "The moral of this story is that simply specifying a social welfare function may not be enough to fully determine a procedure for collective decision making. One must also specify *when* the individuals' preferences or utility levels should be evaluated; before or after the resolution of uncertainties. The timing of social welfare analysis may make a difference. The timing-effect is

[1] See for instance Broome (1984), Diamond (1967) and Myerson (1981) and Hammond (1981), among others, for theoretical work on the timing effect. See Yaari and Bar-Hillel (1984) for empirical evidence about the importance of beliefs in distributional issues.

often an issue in moral debate, as when people argue about whether a social system should be judged with respects to its actual income distribution or with respect to its distribution of economic opportunities" (p. 884).

To the best of our knowledge, the principles that should be followed to answer this question have not yet been identified in the economic literature. Whereas an extensive body of literature exists on inequality measurement when no uncertainty is involved, very little has been written on inequality measurement under uncertainty, with the important exception of Ben Porath, Gilboa and Schmeidler (1997).

As stated by Ben Porath, Gilboa and Schmeidler (1997), the crucial issue for measuring inequality under uncertainty is to simultaneously take into account the inequality of expected incomes and the expected inequalities of actual incomes. In this paper, we propose a simple axiomatic characterization of social welfare rankings under uncertainty that captures these two dimensions.

The paper is organized in the following way. Section 2 introduces notation and provides an axiomatic characterization of social welfare functions under uncertainty. Our most general result is that a small number of reasonable assumptions regarding welfare orderings under uncertainty rule out pure *ex ante* and pure *ex post* evaluations. Any social welfare function that satisfies these axioms should remain *strictly* between the *ex ante* and the *ex post* evaluations of income distributions. Section 3 provides a reasonable strengthening of our basic axioms which leads to a more complete characterization of admissible social welfare functions and Section 4 gives our conclusions.

2 A General Class of Social Preferences Under Uncertainty

In order to better understand the difficulties raised by uncertainty in evaluating income distributions, let us examine the canonical examples given by Ben Porath, Gilboa and Schmeidler (1997). Consider a society with two individuals, a and b, facing two equally likely possible states of the world, s and t, and assume that the planner has to choose among the three following social policies, P_1, P_2 and P_3:

P_1	a	b
s	0	0
t	1	1

P_2	a	b
s	1	0
t	0	1

P_3	a	b
s	1	0
t	1	0

As argued by Ben Porath, Gilboa and Schmeidler (1997), P_2 and P_3 are *ex post* equivalents, since in both cases, whatever the state of the world, the final income distribution is $(0, 1)$ (or $(1, 0)$ which, assuming anonymity, is equivalent). On the other hand, P_3 gives 1 for sure to one individual, and 0 to the other, while P_2 provides both individuals with the same *ex ante* income prospects. On these grounds, for a sufficiently low level of uncertainty aversion, it is reasonable to think that P_2 should be ranked above P_3. As for P_1, on the other hand, both individuals face the same income prospects like in P_2; but in

P_1, there is no *ex post* inequality, whatever the state of the world. This could lead one to prefer[2] P_1 over P_2.

This example makes clear that there is no hope for providing a reasonable social welfare function over income distributions under uncertainty by simply reducing the problem under consideration to a problem of a choice over uncertain aggregated incomes (say, e.g., by computing a traditional social welfare function *à la* Atkinson-Kolm-Sen in each state, and then reducing the problem to a single decision maker's choice among prospects of welfare). Similarly, reducing the problem by first aggregating individuals' income prospects, and then considering a classical social welfare function defined on these aggregated incomes would not be a reasonable solution. The first procedure would lead us to neglect *ex ante* considerations and to judge P_2 and P_3 as equivalent. In contrast, the second procedure would lead us to neglect *ex post* considerations and to see P_1 and P_2 as equivalent. In other words, these procedures would fail to simultaneously take into account the *ex ante* and the *ex post* income distributions.

Ben Porath, Gilboa and Schmeidler (1997) suggest solving this problem by considering a linear combination of the two procedures described above, i.e., a linear combination of the expected Gini index and the Gini index of expected income. This solution captures both *ex ante* and *ex post* inequalities. Furthermore, it is a natural generalization of the principles commonly used for evaluating inequality under certainty on the one hand, and for decision making under uncertainty on the other hand. However, the procedure suggested by Ben Porath, Gilboa and Schmeidler (1997) is not the only possible evaluation principle that takes into account both *ex ante* and *ex post* inequalities. Any functional that is strictly increasing in both individuals' expected income and snapshot inequalities (say, measured by the Gini index) has the same nice property, provided that it takes its values between the expected Gini and the Gini of the expectation. Furthermore, it is unclear why we should restrict ourselves, as Ben Porath, Gilboa and Schmeidler (1997) did, to decision makers who behave in accordance with the multiple priors model[3].

There is hence a need for an axiomatic characterization of inequality measurement under uncertainty, which can encompass the Ben Porath, Gilboa and Schmeidler's (1997) proposal, and make clear why this specific functional should be used. In this section, we propose a set of axioms that capture what we think to be the basic requirements for any reasonable evaluation of welfare under uncertainty, and identify the corresponding general class of preferences.

2.1 Notation

Let $S = \{1, ..., s\}$ and $K = \{1, ..., n\}$ be respectively a finite set of states of the world, and a finite set of individuals. Let $\mathcal{F}$ denote the set of non-negative real-valued functions

[2] As in Gilboa, Ben Porath and Schmeidler (1997), we consider preferences over final allocations: we do not claim that one could not obtain a policy that is strictly preferred to P_1 by way of *ex post* transfers among individuals in P_2

[3] The multiple priors model assumes that social preferences are concave. It is unclear why preferences over uncertain outcomes should necessarily be concave.

on $S \times K$. An element f of $\mathcal{F}$ corresponds to a $(s \times n)$ non-negative real-valued matrix. For every $k \geq 0$, we will denote by $\mathbf{k}$ the $(s \times n)$ matrix with all entries equal to k.

In this paper, we interpret $\mathcal{F}$ as a set of income distributions under uncertainty. For each f in $\mathcal{F}$, $f_{\sigma i}$ denotes i's income if state σ occurs, while $f_{\sigma\cdot}$ is the row vector that represents the income distribution in state σ, and $f_{\cdot i}$ the column vector that represents individual i's income profile. Furthermore, $\mathbf{f}_{\sigma\cdot}$ denotes the $(s \times n)$ matrix with all rows equal to $f_{\sigma\cdot}$, whereas $\mathbf{f}_{\cdot i}$ denotes the $(s \times n)$ matrix with all columns equal to $f_{\cdot i}$. The set of $\mathbf{f}_{\sigma\cdot}$ matrices represents situations where there is no uncertainty: the income distribution is the same in each possible state of the world. In contrast, the $\mathbf{f}_{\cdot i}$ matrices characterize situations where there is no inequality: each individual is faced with the same income prospects.

In the sequel, we adopt the following convention: vectors of $\mathbb{R}^n_+$ and $\mathbb{R}^{n+s}_+$ are considered as row vectors, whereas vectors of $\mathbb{R}^s_+$ are considered as column vectors. For $(x_1, ..., x_p), (y_1, ..., y_p) \in \mathbb{R}^p$, $(x_1, ..., x_p) > (y_1, ..., y_p)$ means that $x_i \geq y_i$ for all i, and there exists at least one j such that $x_j > y_j$.

Finally, we use the following definitions. A function $\phi : \mathbb{R}^p \to \mathbb{R}$ is strictly increasing if for all $x, y \in \mathbb{R}^p$, $x > y$ implies $\phi(x) > \phi(y)$. We say that ϕ is homogeneous if, for all $\theta > 0$, and all $x \in \mathbb{R}^p$, $\phi(\theta x) = \theta\phi(x)$. We say that ϕ is homogeneous of degree 0 if for all $\theta > 0$, and all $x \in \mathbb{R}^p$, $\phi(\theta x) = \phi(x)$. Finally, we say that ϕ is affine if, for all $x \in \mathbb{R}^p$, all $\theta > 0$ and all $\eta \in \mathbb{R}$, $\phi(\theta x + \eta) = \theta\phi(x) + \eta$.

Following the literature on inequality measurement (see, e.g., Atkinson (1970), and Sen (1973)), we do not make any assumptions about individuals' preferences. The issue is not to aggregate individuals' preferences, but to propose principles for defining a reasonable collective attitude towards inequality under uncertainty.

2.2 The structure of social welfare preferences under uncertainty

We assume that there is a complete, continuous preorder on $\mathcal{F}$. This is the usual basic axiom in the field of normative inequality measurement.

Axiom 1 (ORD) *There is a complete, continuous preorder on $\mathcal{F}$, denoted as $\succeq$.*

The preorder $\succeq$ can be interpreted as the decision maker's preference relation over $\mathcal{F}$ (one can see this "decision maker" as anybody behind the veil of ignorance). As usual, $\sim$ and $\succ$ will stand for the symmetric and asymmetric part of $\succeq$, respectively.

Within this framework, we are now going to introduce four axioms which in our view, should be satisfied by any plausible social preference over uncertain income distributions.

The first axiom is a standard monotonicity requirement: if f provides each individual with a higher income than g in each state of the world, then f should be preferred to g.

Axiom 2 (MON) *For all f, g in $\mathcal{F}$, if $f_{\sigma i} > g_{\sigma i}$ for all σ in S and all i in K, then $f \succ g$.*

Any preorder $\succeq$ on $\mathcal{F}$ naturally induces two preorders $\succeq_a$ and $\succeq_p$ on $\mathbb{R}^s_+$ and $\mathbb{R}^n_+$ respectively, defined as: $f_{\sigma\cdot} \succeq_p g_{\sigma\cdot}$ if and only $\mathbf{f}_{\sigma\cdot} \succeq \mathbf{g}_{\sigma\cdot}$, and $f_{\cdot i} \succeq_a g_{\cdot i}$ if and only $\mathbf{f}_{\cdot i} \succeq \mathbf{g}_{\cdot i}$. The preorder $\succeq_p$ captures the decision maker's preferences in the absence of uncertainty, i.e., when the income distribution does not depend on the state of the world. In contrast, $\succeq_a$ captures the decision maker's preferences in the absence of inequality, i.e., when each individual faces the same income prospects. In other words $\succeq_a$ and $\succeq_p$ represent preorders on individual income profiles and snapshot income distributions, respectively.

Let us assume that f and g are such that (a) $f_{\sigma\cdot}$ is preferred to $g_{\sigma\cdot}$ for all σ (with respect to $\succeq_p$), and (b) $f_{\cdot i}$ is preferred to $g_{\cdot i}$ for all i (with respect to $\succeq_a$). In other words, f is preferred to g *ex post* regardless of the state of the world and f is also preferred to g *ex ante* regardless of the individual on which we focus. In such a case, it is reasonable to assume that f is preferred to g with respect to $\succeq$. This property corresponds to the following axiom of dominance.

Axiom 3 (DOM) *Let $f, g \in \mathcal{F}$. If for all $\sigma \in S$, $f_{\sigma\cdot} \succeq_p g_{\sigma\cdot}$, and for all $i \in K$, $f_{\cdot i} \succeq_a g_{\cdot i}$, then $f \succeq g$. If, moreover, there exists $\sigma \in S$ or $i \in K$ such that $f_{\sigma\cdot} \succ_p g_{\sigma\cdot}$ or $f_{\cdot i} \succ_a g_{\cdot i}$, then $f \succ g$.*

(DOM) should not be understood as providing a rule for aggregating individuals' preferences. By construction, $\succeq_a$ does not represent individuals' preferences but the collective attitude towards uncertainty, exactly as $\succeq_p$ represents the collective attitude towards inequality. When these principles imply that (a) any individual is better off in f than in g, and (b) any snapshot distribution of f is better than the corresponding snapshot distribution in g, then (ADOM) simply imposes to prefer f to g.

Now, let us assume that the uncertain income $f_{\sigma i}$ of individual i in state σ can be represented as the combination of individual fixed effects that do not depend on the state of the nature, captured by λ_i, on the one hand, and effects that depend on the state of the nature μ_σ, but that are the same for all individuals, on the other hand. In other words, $f_{\sigma i} = \lambda_i \mu_\sigma$, for all $i \in K$ and all $\sigma \in S$. In such a case, we can reasonably focus on preorders that satisfy the following property: if the distribution of individual (sure) fixed effects is the same for two matrices f and g, but the random variable that generates the variability of individuals' income across states of nature in f is preferred (with respect to $\succeq_a$) to the one that generates the variability of the individuals' income across states of nature in g, then f is preferred to g. This requirement is formally stated in the following Conditional Dominance Axiom.

Axiom 4 (CDOM) $\forall \lambda \in \mathbb{R}^n_+$, $\lambda \neq 0$, $\mu, \nu \in \mathbb{R}^s_+$, $\mu\lambda \succeq \nu\lambda \Leftrightarrow \mu \succeq_a \nu$

Lastly, we will require that $\succeq$ be homogeneous. This axiom is of course debatable[4]; however, this assumption is quite standard in the field of inequality measurement.

Axiom 5 (HOM) $\forall f, g \in \mathcal{F}$, $\forall \theta > 0, f \succeq g \Leftrightarrow \theta f \succeq \theta g$

[4]Homogeneity is a potentially problematic property when there is a positive minimum of subsistance.

The following Lemma will prove to be useful in the sequel.

Lemma 1 *Assume Axioms* (ORD), (CDOM) *and* (HOM) *hold. There then exist a homogeneous function I which represents $\succeq$, and two homogeneous functions I_a and I_p which represent $\succeq_a$ and $\succeq_p$, respectively, such that:*

$$\forall \mu \in \mathbb{R}_+^s,\ \lambda \in \mathbb{R}_+^n,\ I(\mu\lambda) = I_a(\mu) I_p(\lambda).$$

Proof. See the Appendix. ∎

Our first basic finding is that any homogeneous continuous complete social evaluation of the elements of $\mathcal{F}$ that satisfies the dominance and monotonicity axioms introduced in this section should necessarily remain between two very crucial bounds, namely the social evaluation of uncertain inequalities and the social evaluation of unequal uncertainties. In order to state this result, we will need the following notation. After Ben Porath, Gilboa and Schmeidler (1997), for all f in $\mathcal{F}$, and any function $I_a : \mathbb{R}_+^s \to \mathbb{R}$ and $I_p : \mathbb{R}_+^n \to \mathbb{R}$, we will denote by $(I_a * I_p)(f)$ the iterative application of I_a to the results of I_p applied to the rows of f, and by $(I_p * I_a)(f)$ the iterative application of I_p to the results of I_a applied to the columns of f. Formally, we use the following notation: $I_a(f) = (I_a(f_{\cdot 1}), ..., I_a(f_{\cdot n}))$, $I_p(f) = (I_p(f_{1\cdot}), ..., I_p(f_{s\cdot}))$, and $(I_a * I_p)(f) = I_a(I_p(f))$, $(I_p * I_a)(f) = I_p(I_a(f))$. This is a slight abuse in notation, but there is no risk of confusion between $I_a(f_{\cdot i})$ $(I_p(f_{\sigma\cdot}))$, which is a function from $\mathbb{R}_+^s$ $(\mathbb{R}_+^n)$ to $\mathbb{R}$, and I_a (I_p), which is a function from $\mathcal{F}$ to $\mathbb{R}^n$ $(\mathbb{R}^s)$. Our result then reads as follows.

Theorem 1 *Axioms* (ORD), (MON), (DOM) (CDOM) *and* (HOM) *are satisfied if, and only if, there exists a continuous, strictly increasing and homogeneous function Ψ : $\mathbb{R}_+^{s+n} \to \mathbb{R}_+$, two continuous, increasing and homogeneous functions $I_a : \mathbb{R}_+^s \to \mathbb{R}_+$ and $I_p : \mathbb{R}_+^n \to \mathbb{R}_+$, which represent $\succeq_a$ and $\succeq_p$, respectively, such that the following hold:*

1. $\forall f, g \in \mathcal{F},\ f \succeq g \Leftrightarrow I(f) = \Psi(I_p(f), I_a(f)) \geq \Psi(I_p(g), I_a(g)) = I(g)$
2. *If* $(I_a * I_p)(f) = (I_p * I_a)(f)$ *then* $I(f) = \Psi(I_p(f), I_a(f)) = (I_a * I_p)(f)$
3. *If* $(I_a * I_p)(f) \neq (I_p * I_a)(f)$ *then:*

$$\min\{(I_a * I_p)(f), (I_p * I_a)(f)\} < I(f) < \max\{(I_a * I_p)(f), (I_p * I_a)(f)\}.$$

Moreover, Ψ is unique, and I_a and I_p are unique up to an increasing affine transformation.

Proof. See the Appendix. ∎

The symmetry of the representation theorem might at first sight seem surprising, since Axiom (CDOM) is not symmetric. However, once homogeneity is assumed, (CDOM) implies its symmetric counterpart, as Lemma 1 clearly shows. This is stated formally in the following remark.

Remark 1 *Axioms* (ORD), (CDOM) *and* (HOM) *imply that for all* $\lambda, \hat{\lambda} \in \mathbb{R}^n_+$, $\mu \in \mathbb{R}^s_+$, $\mu \neq 0$, $\mu\lambda \succeq \mu\hat{\lambda} \Leftrightarrow \lambda \succeq_p \hat{\lambda}$

The I_a function represents $\succeq_a$ and reflects how the decision maker evaluates uncertain income profiles. Symmetrically, the I_p function represents $\succeq_p$ and captures how the decision maker evaluates income distribution under certainty. Within this framework, $(I_a * I_p)$ represents the evaluation through I_a of the uncertain social welfare profiles, while $(I_p * I_a)$ represents the evaluation through I_p of the distribution of income profiles across individuals. These two functionals represent the two key dimensions of social welfare under uncertainty, namely, unequal uncertainties $(I_p * I_a)$ and uncertain inequalities $(I_a * I_p)$. The first one reflects *ex post* considerations while the second one only captures *ex ante* considerations. Theorem 1 shows that under plausible monotonicity and dominance assumptions, a continuous and homogeneous social evaluation cannot correspond to $(I_p * I_a)$ or $(I_a * I_p)$, but should necessarily remain strictly between these two bounds.

The social welfare functionals defined in Theorem 1 are such that for every f, $I(f)$ is a specific weighted-average[5] of the iterative application of I_p to the results of I_a and of the iterative application of I_a to the results of I_p. This motivates the following definition of *Weighted Cross-Iterative* (WCI) functionals.

Definition 1 *A continuous functional* $I : \mathcal{F} \to \mathbb{R}$ *is a* Weighted Cross-Iterative *(WCI) functional, if and only if, there exist two continuous, increasing and homogeneous functions* $I_a : \mathbb{R}^s_+ \to \mathbb{R}$ *and* $I_p : \mathbb{R}^n_+ \to \mathbb{R}$, *a function* $\gamma : \mathcal{F} \to (0, 1)$ *homogeneous of degree 0, such that the following hold:*
(i) $\forall f \in \mathcal{F}$, $I(f) = \gamma(f)(I_p * I_a)(f) + (1 - \gamma(f))(I_a * I_p)(f)$
(ii) $\forall f, g \in \mathcal{F}$, $(I_a(f), I_p(f)) > (I_a(f), I_p(g)) \Rightarrow I(f) > I(g)$.
We denote $\mathcal{W}$ *as the set of WCI functionals.*

Using definition 1, theorem 1 can be restated as follows.

Theorem 2 *Axioms* (ORD), (MON), (DOM), (CDOM) *and* (HOM) *are satisfied if, and only if,* $\succeq$ *can be represented by* $I \in \mathcal{W}$, *with* I_a *and* I_p *representing* $\succeq_a$ *and* $\succeq_p$, *respectively. Moreover,* I_a *and* I_p *are unique up to an increasing affine transformation, and* $\gamma_{|\{f \in \mathcal{F} | (I_a * I_p)(f) \neq (I_p * I_a)(f)\}}$ *is unique.*

Proof. See the Appendix. ■

Note that the functionals proposed by Ben Porath, Gilboa and Schmeidler (1997), namely $I(f) = \alpha(G * E)(f) + (1 - \alpha)(E * G)(f)$, where E is the expectation and G is a Gini functional, belong to $\mathcal{W}$. Of course, the class of WCI functionals is much larger, since WCI functionals do not necessarily give constant weights to uncertainty in social welfare, on the one hand, and to inequality in uncertain income profiles, on the other

[5] To be more specific, for each f, there exists $\gamma(f)$ in $(0, 1)$, such that $I(f) = \gamma(f)\ (I_p * I_a)(f) + (1 - \gamma_(f))(I_a * I_p)(f)$.

hand. Actually, the most striking feature of the functionals proposed by Ben Porath, Gilboa and Schmeidler (1997) is precisely that these relative weights do not depend on the matrix f under consideration (they are always given by the same α and $(1-\alpha)$).

Interestingly, theorem 1 can be used to derive a very fundamental *partial ordering* over distributions of income under uncertainty: for any $f, g \in \mathcal{F}$, if

$$\max\left\{(I_a * I_p)(f), (I_p * I_a)(f)\right\} \leq \min\left\{(I_a * I_p)(g), (I_p * I_a)(g)\right\},$$

then $g \succeq f$. If f exhibits both less uncertainty in social welfare and less inequality in uncertain profiles than g, then it should be preferred to g.

This result provides a very simple means for ranking a wide range of distributions of income under uncertainty. For instance, consider the three social policies P_1, P_2 and P_3 defined at the beginning of this section. Assuming[6] that $I_p(1,...,1) = I_a(1,...,1) = 1$, we can easily check that $(I_a * I_p)(P_1) = (I_p * I_a)(P_2) = I_a(0,1)$, $(I_a * I_p)(P_3) = (I_p * I_a)(P_3) = I_p(0,1)$, $(I_a * I_p)(P_2) = I_p(0,1)$ and $(I_p * I_a)(P_2) = I_a(0,1)$. Therefore, only two cases are possible: $P_1 \succ P_2 \succ P_3$ or $P_3 \succ P_2 \succ P_1$. Which of these orderings holds depends on the relative weight of the inequality and uncertainty aversions. If we assume that I_a is the expectation, and I_p the Gini index, we get $P_1 \succ P_2 \succ P_3$. This is so because the expectation is neutral towards risk.

3 Weighted Cross-Iterative Functionals

In this section, we show that a reasonable strengthening of the requirements introduced in the previous section makes it possible to characterize interesting and easy-to-implement sub-classes within the set of WCI functionals. Therefore, hereafter, we assume that $\succeq$ can be represented by a WCI functional.

First, we are going to focus on WCI functionals that satisfy the following strengthening of (DOM), to which we refer to as an Average Dominance Axiom[7].

Axiom 6 (ADOM) *$\forall f, g \in \mathcal{F}$, if $(I_a * I_p)(f) \geq (I_a * I_p)(g)$ and $(I_p * I_a)(f) \geq (I_p * I_a)(g)$ then $I(f) \geq I(g)$.*

This Axiom corresponds to requirements that are clearly stronger than (DOM). Under (ADOM), we do not require uniform *ex ante* and *ex post* dominance to prefer f to g, but only average dominance. Axiom (ADOM) can be seen as an axiom that imposes some consistency in the principles that rule *ex post* and *ex ante* welfare evaluations. To compare two matrices from an *ex post* viewpoint, we must first evaluate each possible income distribution and then, in a second stage, compare the two sets of social welfare evaluations. Symmetrically, to compare two matrices from an *ex ante* viewpoint, we must first evaluate income profiles for each individual, and then, in a second stage, compare the two distributions of income profile evaluations. In a sense, axiom (ADOM) says that

[6] Which is only a matter of normalization.

[7] Note that for any WCI functional, I, I_a and I_p are well-defined.

the principles that rule the first stage of the *ex post* comparison should be the same as those which rule the second stage of the *ex ante* comparison, and vice versa. To put it differently, since each possible income distribution is evaluated through I_p, the distribution of income profiles should also be evaluated through I_p. Symmetrically, since each individual's income profile is evaluated through I_a, the social welfare evaluation profiles should also be evaluated through I_a.

In addition to axiom (ADOM), we will require $\succeq$ to be additive, meaning that adding the same intercept to two matrices does not modify their ranking.

Axiom 7 (ADD) *For all* $f, g \in \mathcal{F}$, $\eta \in \mathbb{R}_+$, $f \succeq g \Rightarrow f + \eta \mathbf{1} \succeq g + \eta \mathbf{1}$

This is a standard assumption regarding social welfare orderings. We could have introduced (ADD) earlier in the text. To be more specific, we could have introduced (ADD) instead of (HOM) in the previous section: substituting (ADD) for (HOM) in the list of axioms used in theorem 1 leads to the same general class of social preferences (with some obvious modifications).

The following theorem characterizes WCI functionals which satisfy (ADOM) and (ADD).

Theorem 3 $\succeq$ *can be represented by a WCI functional* I*, and satisfies Axioms* (ADOM) *and* (ADD) *if, and only if,* I_a *and* I_p *are affine, and there exist* $\alpha, \beta \in (0,1)$*, such that:*

$$I(f) = \begin{cases} \alpha\,(I_p * I_a)(f) + (1-\alpha)\,(I_a * I_p)(f), & \text{if } (I_a * I_p)(f) \geq (I_p * I_a)(f) \\ \beta\,(I_p * I_a)(f) + (1-\beta)\,(I_a * I_p)(f), & \text{if } (I_p * I_a)(f) \geq (I_a * I_p)(f) \end{cases}$$

Moreover, α *and* β *are unique. The set of such* I *is denoted* $\mathcal{W}_1$*.*

Proof. See the Appendix. ■

Once (ADOM) and (ADD) are satisfied, the weight given to *ex ante* evaluations only depends on whether they are more important or less important than *ex post* ones, and *vice versa.*

Axiom (ADOM) can be strenghtened by assuming that the two fundamental dimensions of welfare, namely inequality in uncertainties and uncertainty in inequalities are of commensurate value and equally important. The following axiom of Global Dominance requires that if the best dimension of a matrix f is better than the best dimension of a matrix g, and the worst dimension of f is also better than the worst dimension of g, then f is better than g.

Axiom 8 (GDOM) *For all* f, g *in* $\mathcal{F}$*, if*

$$\begin{cases} \max\{(I_a * I_p)(f), (I_p * I_a)(f)\} \geq \max\{(I_a * I_p)(g), (I_p * I_a)(g)\} \\ \min\{(I_a * I_p)(f), (I_p * I_a)(f)\} \geq \min\{(I_a * I_p)(g), (I_p * I_a)(g)\} \end{cases}$$

then, $I(f) \geq I(g)$*.*

Note that when (GDOM) is satisfied, (ADOM) is also satisfied.

Replacing Axiom (ADOM) by Axiom (GDOM) in Theorem 3 leads to the characterization of the *Weighted Max-Min* functionals, i.e., of WCI functionals that can be written as a weighted average of the maximum and the minimum of $(I_p * I_a)(f)$ and $(I_a * I_p)(f)$.

Theorem 4 *$\succeq$ can be represented by a WCI functional I, and satisfies Axioms* (GDOM) *and* (ADD) *if, and only if, I_a and I_p are affine and there exists $\delta \in (0,1)$, such that:*

$$I(f) = \delta \min \{(I_a * I_p)(f), (I_p * I_a)(f)\} + (1-\delta) \max \{(I_a * I_p)(f), (I_p * I_a)(f)\}$$

Moreover, δ is unique. The set of such I is denoted as $\mathcal{W}_2$.

Proof. See the Appendix. ■

Once (GDOM) is satisfied, the weights put on the two possible welfare evaluations do not depend on whether they correspond to *ex post* or *ex ante* considerations, but only on whether they are the most or the least important. Observe that we have $\mathcal{W}_2 \subset \mathcal{W}_1 \subset \mathcal{W}$.

Weighted Cross-Iterative Functionals and Ben-Porath, Gilboa, Schmeidler's proposal

As noted above, Ben Porath, Gilboa and Schmeidler (1997) have proposed a specific sub-class of WCI functionals, namely the functionals that can be written as $I(f) = \alpha(I_p * I_a)(f) + (1-\alpha)(I_a * I_p)(f)$, where I_a and I_p are what they call min-of-means functionals.

Min-of-means functionals are well-known in decision theory under the name of the multiple priors model, and were first introduced by Gilboa and Schmeidler (1989). Special notation is needed in order to define these functionals. Let $\mathcal{P}_K$ and $\mathcal{P}_S$ be the spaces of probability vectors on K and S, respectively. For any $f_{\sigma\cdot} \in \mathbb{R}^n_+$ and $q \in \mathcal{P}_K$, let $q \cdot f_{\sigma\cdot} = \sum_i q_i f_{\sigma i}$. Similarly, for any $f_{\cdot i} \in \mathbb{R}^s_+$, and $q \in \mathcal{P}_S$, let $q \cdot f{\cdot}i = \sum_\sigma q_\sigma f_{\sigma i}$. Min-of-means functionals are defined as follows.

Definition 2 *A functional $I_a : \mathbb{R}^s_+ \to \mathbb{R}$ ($I_p : \mathbb{R}^n_+ \to \mathbb{R}$) is a* symmetric min-of-means *functional if, and only if, there exists a unique compact symmetric*[8] *and convex subset $\mathcal{C}_{I_a}$ ($\mathcal{C}_{I_p}$) of $\mathcal{P}_S$ ($\mathcal{P}_K$), such that for all $f_{\cdot i} \in \mathbb{R}^s_+$ ($f_{\sigma\cdot} \in \mathbb{R}^n_+$), $I_a(f_{\cdot i}) = \min\limits_{q \in \mathcal{C}_{I_a}} q \cdot f_{\cdot i}$ ($I_p(f_{\sigma\cdot}) = \min\limits_{q \in \mathcal{C}_{I_p}} q \cdot f_{\sigma\cdot}$).*

The class of functionals $\mathcal{W}_3$ proposed by Ben-Porath, Gilboa and Schmeidler can then be defined as:

$$\mathcal{W}_3 = \{I \in \mathcal{W} \,|I_a, I_p \text{ are symmetric min-of-means functionals and } \forall f, g \in \mathcal{F},\ \gamma(f) = \gamma(g)\}$$

[8]We say that a subset $\mathcal{C}$ of $\mathcal{P}_\mathcal{S}$ ($\mathcal{P}_\mathcal{K}$) is symmetric if, and only if, for all $\mu \in C$, any μ' obtained by a permutation of the components of μ also belongs to C.

Any functional in $\mathcal{W}_3$ clearly belongs to $\mathcal{W}_1$, meaning $\mathcal{W}_3 \subset \mathcal{W}_1$. In contrast, elements of $\mathcal{W}_3$ do not necessarily satisfy (GDOM), and there exist functionals in $\mathcal{W}_3$ which do not belong to $\mathcal{W}_2$ (i.e., $\mathcal{W}_3 \not\subset \mathcal{W}_2$).

Any I in $\mathcal{W}_3$ gives the same weight to *ex ante* inequalities regardless of whether they are more or less important than *ex post* ones and vice versa. In contrast, any I in $\mathcal{W}_2$ systematically puts more emphasis on the dominant source of inequality. The following theorem explores the relationships between these two classes of functionals.

Theorem 5 $I \in \mathcal{W}_2 \cap \mathcal{W}_3$ *if, and only if, one of the two following conditions are satisfied:*
(i) $\forall f \in \mathcal{F}$, $I(f) = \frac{1}{2}(I_p * I_a)(f) + \frac{1}{2}(I_a * I_p)(f)$
(ii) I_a *or* I_p *are either the mathematical expectation or the minimum operator.*

Proof. See the Appendix. ■

Interestingly, once we exclude the specific cases of risk (or inequality) neutrality and extreme egalitarianism (or extreme aversion to risk), the only functionals that belong simultaneously to $\mathcal{W}_2$ and $\mathcal{W}_3$ are the arithmetic means of *ex ante* and *ex post* social welfare evaluation (through min-of-means). The key feature of these functionals is that any given shifts in *ex post* levels of social welfare can actually be compensated by symmetric shifts in *ex ante* levels of individuals' welfare, i.e., by shifts whose social evaluation is the same as the evaluation of the *ex post* shifts. To make this property explicit, let us define, for each vector u in $\mathbb{R}^n$, the set $\mathcal{S}(u)$ of vectors v of $\mathbb{R}^s$, such that for some constant $k > 0$, the matrix with all rows equal to[9] $u + k_{\sigma\cdot}$ (i.e., a matrix with no uncertainty, where only inequality matters) is equivalent to a matrix with all columns equal to $v + k_{\cdot i}$ (i.e., a matrix with no inequality, where only uncertainty matters). Formally:

$$\mathcal{S}(u) = \left\{ v \in \mathbb{R}^s \,|\, \exists k > 0, \text{ s.t. } \mathbf{v}_{\cdot i} + \mathbf{k} \sim \mathbf{u}_{\sigma\cdot} + \mathbf{k} \right\}.$$

Then, for any matrix $f \in \mathcal{F}$, one can define the set $\mathcal{E}(f) \subseteq \mathcal{F}$ of matrices that are obtained from f by shifts in *ex post* levels of social welfare and shifts in *ex ante* levels of individuals' welfare, whose social evaluations are the same. Formally,

$$\mathcal{E}(f) = \{ g \in \mathcal{F} \,|\, \exists u \in \mathbb{R}^n,\ v \in \mathcal{S}(u), \text{ s.t. } (I_a(g), I_p(g)) = (I_a(f) - u, I_p(f) + v) \}$$

We can now state formally the desired Axiom of symmetry.

Axiom 9 (SYM) $\forall f \in \mathcal{F}$, $g \in \mathcal{E}(f) \Rightarrow f \sim g$.

As it turns out, the preorder $\succeq$ can be represented by a WCI functional and satisfies Axioms (ADD) and (SYM) if, and only if, it can be represented by the arithmetic mean of *ex ante* and *ex post* welfare evaluations.

[9] We denote by $k_{\sigma\cdot}$ the vector of $\mathbb{R}^n$ with all entries equal to k, and by $k_{\cdot i}$ the vector of $\mathbb{R}^s$ with all entries equal to k.

Theorem 6 *$\succeq$ can be represented by a WCI functional I, and satisfies Axioms (ADD) and (SYM) if, and only if, I_a and I_p are affine, and:*

$$I(f) = \frac{1}{2}(I_p * I_a)(f) + \frac{1}{2}(I_a * I_p)(f).$$

Proof. See the Appendix. ■

4 Conclusion

In this paper, we show that under some reasonable monotonicity and dominance assumptions, any continuous homogeneous social welfare function should lie strictly between the *ex ante* and the *ex post* evaluations of income distributions. We propose the weighted average of the minimum and the maximum of *ex post* and *ex ante* evaluations as a new means for evaluating welfare under uncertainty.

Clearly, this new evaluation tool can be used in a potentially very large set of contexts. The usual practice is to rank public policies according to their impact on either the observed distribution of income or on the distribution of expected income. Once we do not neglect macroeconomic uncertainty, we should not rely on either pure *ex ante* or pure *ex post* considerations, but on one of the mixtures that are axiomatized in this paper.

At a very general level, our paper can be understood as an attempt to evaluate income distributions when it is not indifferent whether income varies across states of the world or across individuals. We think that this approach could be generalized to any problem of welfare evaluation where the *sources* of income variability matter. One such problem is the evaluation of income distributions according to the principle of equality of opportunity. This principle requires to give different weights to inequalities generated by circumstances beyond the control of individuals on the one hand, and on the other hand, to inequalities generated by actions that reflect individuals' own free volition. We speculate that the axiomatization and design of new means for implementing this principle can be obtained following a very similar route as the one used in this paper. This issue is part of our research agenda.

Appendix

Proof of Lemma 1.

By Debreu (1959), Axiom (ORD) implies that there exits a continuous function $I : \mathcal{F} \to \mathbb{R}_+$ representing $\succeq$. We can, therefore, define I_a and I_p, representing $\succeq_a$ and $\succeq_p$, respectively, as follows: $I_a(f_{\cdot i}) = I(\mathbf{f}_{\cdot\mathbf{i}})$, and $I_p(f_{\sigma\cdot}) = I(\mathbf{f}_{\sigma\cdot})$, for all $f_{\cdot i} \in \mathbb{R}^s_+$ and $f_{\sigma\cdot} \in \mathbb{R}^n_+$. Furthermore, we can, without loss of generality, normalize I such that $I(\mathbf{1}) = 1$. Axiom (HOM) implies that I, I_a and I_p are homogeneous.

Let $f = \mu\lambda \in \mathcal{F}$, with $\mu \in \mathbb{R}^s_+$ and $\lambda \in \mathbb{R}^n_+$, $\lambda \neq 0$. Define g by: $g_{\sigma i} = I_a(\mu)\lambda_i$, for all $\sigma \in S$ and all $i \in K$. Observe that $g = \nu\lambda$, with $\nu = (I_a(\mu), ..., I_a(\mu)) \in \mathbb{R}^s_+$. By homogeneity of I_a, and given the normalization choice $I_a(\mathbf{1}) = 1$, we have: $I_a(\nu) = I_a(\mu)$. Therefore, by Axiom (CDOM), we have $f \sim g$, i.e., $I(f) = I(g)$. But, by homogeneity of I, $I(g) = I_a(\mu)I(\mathbf{h}_{\sigma\cdot})$, with $h_{\sigma\cdot} = \lambda$ for every σ. Since, by definition of I_p, $I(\mathbf{h}_{\sigma\cdot}) = I_p(\lambda)$, we get: $I(g) = I_a(\mu)I_p(\lambda) = I(f)$, the desired result. ■

Proof of Theorem 1.

First, we prove the "only if" part of the Theorem.

By Debreu (1959), Axiom (ORD) holds if, and only if, there exists a continuous function $I : \mathcal{F} \to \mathbb{R}$ such that I represents $\succeq$. Furthermore, Axiom (MON) implies that I is increasing.

Without loss of generality, we can choose I such that $I(\theta\mathbf{1}) = \theta$ for all $\theta > 0$. Then, Axiom (HOM) implies that I is homogeneous, i.e., $I(\theta f) = \theta I(f)$ for all $\theta > 0$ and $f \in \mathcal{F}$.

Considering the restriction of I on sets of matrices $\mathbf{f}_{\sigma\cdot}$ and $\mathbf{f}_{\cdot\mathbf{i}}$ respectively, Axiom (ORD) implies that there exist two continuous functions I_p and I_a representing $\succeq_p$ and $\succeq_a$ respectively, and that these functions are increasing and homogeneous, since I is. Furthermore, Axiom (DOM) implies that I is separable in the following sense: there exists a continuous, strictly increasing function $\Psi : \mathbb{R}^{s+n}_+ \to \mathbb{R}$ such that $I(f) = \Psi\left(I_p(f), I_a(f)\right)$. Since I, I_p and I_a are homogeneous, so is Ψ.

Now, let $f \in \mathcal{F}$, with $f \neq \mathbf{0}$ and define[10] g and h as follows: $g_{\sigma i} = \frac{I_a(f_{\cdot i})I_p(f_{\sigma\cdot})}{(I_a * I_p)(f)}$ and $h_{\sigma i} = \frac{I_a(f_{\cdot i})I_p(f_{\sigma\cdot})}{(I_p * I_a)(f)}$, for all σ in S and all i in K. Observe that, since $f \neq \mathbf{0}$, Axiom (MON) implies that g and h are well defined.

First, let us assume that $(I_a * I_p)(f) < (I_p * I_a)(f)$. We get: $I_a(g_{\cdot i}) = \frac{I_a(f_{\cdot i})}{(I_a * I_p)(f)}(I_a * I_p)(f)$ for all i in K by homogeneity of I_a. Therefore: $I_a(g_{\cdot i}) = I_a(f_{\cdot i})$ for all i in K. On the other hand, $I_p(g_{\sigma\cdot}) = \frac{I_p(f_{\sigma\cdot})}{(I_a * I_p)(f)}(I_p * I_a)(f)$ for all i in K, by homogeneity of I_p, which implies $I_p(g_{\sigma\cdot}) > I_p(f_{\sigma\cdot})$ for all σ in S, since $(I_a * I_p)(f) < (I_p * I_a)(f)$. Therefore, Axiom (DOM) implies $g \succ f$.

Observe that $g = \frac{1}{(I_a * I_p)(f)}\mu_1\lambda_1$, with $\lambda_1 = (I_a(f_{\cdot 1}), ..., I_a(f_{\cdot n}))$ and $\mu_1 = (I_p(f_{1\cdot}), ..., I_p(f_{s\cdot}))$. Therefore, by homogeneity of I, and using Lemma 1, we have $I(g) = \frac{1}{(I_a * I_p)(f)}I_a(\mu_1)I_p(\lambda_1)$.

[10] Observe that $I(\mathbf{0}) = (I_a * I_p)(\mathbf{0}) = (I_p * I_a)(\mathbf{0}) = 0$, and therefore, condition (2) of the Theorem is obviously satisfied in this case.

By definition, $I_a(\mu_1) = (I_a{*}I_p)(f)$, and $I_p(\lambda_1) = (I_p{*}I_a)(f)$. Therefore, $I(g) = (I_p{*}I_a)(f)$, which implies: $I(f) < (I_p * I_a)(f)$.

On the other hand, $I_a(h_{\cdot i}) = \frac{I_a(f_{\cdot i})}{(I_p * I_a)(f)}(I_a * I_p)(f)$ for all i in K, by homogeneity of I_a. Therefore, $I_a(h_{\cdot i}) < I_a(f_{\cdot i})$ for all i in K since $(I_a * I_p)(f) < (I_p * I_a)(f)$, and $I_p(h_{\sigma \cdot}) = \frac{I_p(f_{\sigma \cdot})}{(I_p * I_a)(f)}(I_p * I_a)(f)$ for all i in K, by homogeneity of I_p, which implies $I_p(h_{\sigma \cdot}) = I_p(f_{\sigma \cdot})$ for all σ in S. Therefore, by Axiom (DOM), $f \succ h$.

By homogeneity of I, $I(h) = \frac{1}{(I_p * I_a)(f)} I(\lambda_1 \mu_1)$. Therefore, using again Lemma 1, we get: $I(h) = (I_a * I_p)(f)$. Therefore, $I(f) > (I_a * I_p)(f)$. From which it follows that $(I_a * I_p)(f) < I(f) < (I_p * I_a)(f)$.

Using a symmetrical argument, we can show that if $(I_a * I_p)(f) > (I_p * I_a)(f)$, then $(I_p * I_a)(f) < I(f) < (I_a * I_p)(f)$.

Now, let us assume that $(I_a{*}I_p)(f) = (I_p{*}I_a)(f)$. We then clearly get that $f \sim g \sim h$, and therefore, $(I_a * I_p)(f) = (I_p * I_a)(f) = I(f)$.

We will now prove the uniqueness of I_a and I_p up to an increasing affine transformation. Due to the symmetry of the problem, we will focus on I_a (the proof for I_p is similar).

Let us assume that there exist two functionals I_a and $\hat{I}_a$ that represent $\succeq_a$. Let $\tilde{I}_a = \frac{I_a(1,...,1)}{\hat{I}(1,...,1)} \hat{I}_a$. Then, $\tilde{I}_a(1, ..., 1) = I_a(1, ..., 1)$. Assume there exists $\mu \in \mathbb{R}^s_+$ such that $I_a(\mu) \neq \tilde{I}_a(\mu)$. Without loss of generality, let $\tilde{I}_a(\mu) = \xi > I_a(\mu) = \zeta$.

Let us consider $\mu_1 = (\frac{\xi}{I_a(1,...,1)}, ..., \frac{\xi}{I_a(1,...,1)})$. By definition of $\tilde{I}_a$, $\tilde{I}_a(\mu_1) = \frac{I_a(1,...,1)}{\hat{I}_a(1,...,1)} \hat{I}_a(\mu_1)$. the homogeneity of $\hat{I}_a$ then implies: $\tilde{I}_a(\mu_1) = \xi$. Therefore, $\mu \sim_a \mu_1$.

Similarly, let us define $\mu_2 = (\frac{\zeta}{I_a(1,...,1)}, ..., \frac{\zeta}{I_a(1,...,1)})$. Using Axiom (HOM) again, one gets $I_a(\mu_2) = \zeta$. Hence, $\mu_2 \sim_a \mu$.

Since $\mu_1 \sim_a \mu$ and $\mu_2 \sim_a \mu$, we finally get $\mu_1 \sim_a \mu_2$, which contradicts Axiom (MON), since $\xi > \zeta$.

Now, let us turn to the uniqueness of Ψ. Let us assume that there exist two functionals Ψ_1 and Ψ_2, such that $\Psi_1(I_a, I_p)$ and $\Psi_2(I_a, I_p)$ both represent $\succeq$. Since I_a and I_p are defined up to an increasing affine transformation, we can assume without loss of generality that $I_a(1, ..., 1) = I_p(1, ..., 1) = 1$. Let $\Psi_3 = \frac{\Psi_1(1,...,1)}{\Psi_2(1,...,1)} \Psi_2$. Then, $\Psi_3(1, ..., 1) = \Psi_1(1, ..., 1)$. Let us assume there exists f in $\mathcal{F}$, such that $\Psi_3(I_a(f), I_p(f)) \neq \Psi_1(I_a(f), I_p(f))$. Without loss of generality, let $\Psi_3(I_a(f), I_p(f)) = \xi > \Psi_1(I_a(f), I_p(f)) = \zeta$.

Now, let us define g as follows: $g_{\sigma i} = \frac{\xi}{\Psi_1(1,...,1)}$ for all σ and all i. Due to the homogeneity of Ψ_3, and given the normalization of I_a and I_p, $\Psi_3(I_a(g), I_p(g)) = \xi$. Therefore, $g \sim f$.

Similarly, let h be defined by: $h_{\sigma i} = \frac{\zeta}{\Psi_1(1,...,1)}$ for all σ and all i. Due to the homogeneity of Ψ_1, and given the normalization of I_a and I_p, $\Psi_1(I_a(f), I_p(f)) = \zeta$. Therefore, $h \sim f$.

Since $g \sim f$ and $h \sim f$, we finally get $g \sim f$, which contradicts Axiom (MON), since $\xi > \zeta$. Therefore, Ψ is at least unique up to an increasing affine transformation.

Hence, there exist $a > 0$ and $b \in \mathbb{R}$, such that $\Psi_2 = a\Psi + b$. But let us consider $f \in \mathcal{F}$ such that $(I_a * I_p)(f) = (I_p * I_a)(f)$. Such an f exists since we know from

Lemma 1 that for all $\mu \in \mathbb{R}_+^s$ and all $\lambda \in \mathbb{R}_+^n$, $I(\mu\lambda) = I_a(\mu)I_p(\lambda)$. Then, one must have $\Psi_2(I_a(f), I_p(f)) = \Psi_1(I_a(f), I_p(f)) = (I_a * I_p)(f)$. Therefore, $a = 1$ and $b = 0$, which completes the proof of the uniqueness of Ψ.

We will now turn to the "if" part of the theorem.

Axiom (ORD) is obviously satisfied. Since Ψ, I_a and I_p are homogeneous, Axiom (HOM) is satisfied. Furthermore, since Ψ is strictly increasing, Axiom (DOM) holds, and since I_a and I_p are increasing, Axiom (MON) holds too.

Now, let $f = \mu\lambda$ and $g = \nu\lambda$ as in Axiom (CDOM), with $\mu \succeq_a \nu$. We have, $I_a(f_{\cdot i}) = I_a(\mu)\lambda_i$ for all $i \in K$ and $I_p(f_{\sigma\cdot}) = I_p(\lambda)\mu_\sigma$ for all $\sigma \in S$. Therefore, by homogeneity of I_a and I_p, we have $(I_a * I_p)(f) = (I_p * I_a)(f) = I_a(\mu)I_p(\lambda)$. Hence, by condition (1) in the Theorem, it follows that $I(f) = I_a(\mu)I_p(\lambda)$. Similarly, $I(g) = I_a(\nu)I_p(\lambda)$. Therefore, $I(f) \geq I(g)$, if and only if, $I_a(\mu) \geq I_a(\nu)$, i.e., $\mu \succeq_a \nu$: Axiom (CDOM) is hence satisfied.

Finally, any increasing affine transformation of I_a and I_p also leads to a functional representing $\succeq$, which completes the proof. ■

Proof of Theorem 2.

Assume that I satisfies the conditions of Theorem 1. Then, for all f in $\mathcal{F}$, we can define:

$$\begin{cases} \gamma(f) = \frac{I(f)-(I_a*I_p)(f)}{(I_p*I_a)(f)-(I_a*I_p)(f)} \text{ if } (I_p * I_a)(f) \neq (I_a * I_p)(f) \\ \gamma(f) = \frac{1}{2} \text{ if } (I_p * I_a)(f) = (I_a * I_p)(f). \end{cases}$$

Clearly $\gamma(f)$ belongs to $(0,1)$, is homogenous of degree zero and satisfies,

$$I(f) = \gamma(f)(I_p * I_a)(f) + (1 - \gamma(f))(I_a * I_p)(f), \forall f \in \mathcal{F}.$$

Furthermore, condition (1) in Theorem 1 and the requirement that Ψ be strictly increasing implies that for all f, g in $\mathcal{F}$, such that $(I_a(f), I_p(f)) > (I_a(f), I_p(g))$, $I(f) > I(g)$, i.e., condition (ii) of Definition 1 is satisfied. Therefore, if I satisfies the conditions of Theorem 1, it can be written as a WCI functional.

Uniqueness up to an increasing affine transformation of I_a and I_p are proven as in Theorem 1. The uniqueness of $\gamma_{|\{f\in\mathcal{F}|(I_a*I_p)(f)\neq(I_p*I_a)(f)\}}$ is straightforward.

Conversely, any WCI functional with I_a and I_p, representing respectively $\succeq_a$ and $\succeq_p$, obviously satisfies the conditions imposed on I in Theorem 1. ■

Proof of Theorem 3.

The "if" part of the Theorem is straightforward. We hence only prove the "only if" part.

By definition, if I is a WCI, there exists a function $\gamma : \mathcal{F} \to (0,1)$ homogeneous of degree 0, and two homogeneous increasing functions I_a and I_p, which represent respectively $\succeq_a$ and $\succeq_p$, such that $\succeq$ can be represented by $I(f) = \gamma(f)(I_p*I_a)(f)+(1-\gamma(f))(I_a*I_p)(f)$. Without loss of generality, we can normalize I such that $I(\mathbf{1}) = 1$.

The proof goes through three steps.

Claim 1. $\forall f \in \mathcal{F}$ such that $(I_p*I_a)(f) \neq (I_a*I_p)(f)$, $\forall \theta > 0$, $\forall \eta \in \mathbb{R}_+$, $\gamma(\theta f+\eta\mathbf{1}) = \gamma(f)$.

Proof.

For any $f \in \mathcal{F}$, $\theta > 0$, $\eta \in \mathbb{R}_+$, the homogeneity of I and the Axiom (ADD) imply: $I(\theta f + \eta \mathbf{1}) = \theta I(f) + \eta$. Similarly, $I_a(\theta f + \eta \mathbf{1}) = \theta I_a(f) + (\eta, ..., \eta)$, $I_p(\theta f + \eta \mathbf{1}) = \theta I_p(f) + (\eta, ..., \eta)$. Therefore, we can write:

$$
\begin{aligned}
I(\theta f + \eta \mathbf{1}) &= \gamma(\theta f + \eta \mathbf{1})(I_p * I_a)(\theta f + \eta \mathbf{1}) + (1 - \gamma(\theta f + \eta \mathbf{1}))(I_a * I_p)(\theta f + \eta \mathbf{1}) \\
&= \gamma(\theta f + \eta \mathbf{1}) \left[\theta (I_p * I_a)(f) + \eta\right] + (1 - \gamma(\theta f + \eta \mathbf{1})) \left[\theta (I_a * I_p)(f) + \eta\right] \\
&= \theta \left[\gamma(\theta f + \eta \mathbf{1})(I_p * I_a)(f) + (1 - \gamma(\theta f + \eta \mathbf{1}))(I_a * I_p)(f)\right] + \eta. \\
&= \theta \left[\gamma(f)(I_p * I_a)(f) + (1 - \gamma(f))(I_a * I_p)(f)\right] + \eta
\end{aligned}
$$

We can also write, however:

$$I(\theta f + \eta \mathbf{1}) = \theta I(f) + \eta = \theta \left[\gamma(f)(I_p * I_a)(f) + (1 - \gamma(f))(I_a * I_p)(f)\right] + \eta.$$

Comparing the two expressions of $I(\theta f + \eta \mathbf{1})$, we obtain:

$$\theta \left[\gamma(f)(I_p * I_a)(f) + (1 - \gamma(f))(I_a * I_p)(f)\right] + \eta = \theta \left[\gamma(f)(I_p * I_a)(f) + (1 - \gamma(f))(I_a * I_p)(f)\right] + \eta$$

Assuming $(I_p * I_a)(f) \neq (I_a * I_p)(f)$, this implies $\gamma(\theta f + \eta \mathbf{1}) = \gamma(f)$. $\diamondsuit$

Claim 2. Let $f, g \in \mathcal{F}$. If $(I_a * I_p)(g) = (I_a * I_p)(f)$, $(I_p * I_a)(f) = (I_p * I_a)(g)$, and $(I_a * I_p)(f) \neq (I_p * I_a)(f)$ then $\gamma(f) = \gamma(g)$.

Proof.

By Axiom (ADOM), if $(I_a * I_p)(g) = (I_a * I_p)(f)$ and $(I_p * I_a)(f) = (I_p * I_a)(g)$, then $I(f) = I(g)$. Therefore: $\gamma(f)(I_p * I_a)(f) + (1 - \gamma(f))(I_a * I_p)(f) = \gamma(g)(I_p * I_a)(g) + (1 - \gamma(g))(I_a * I_p)(g)$, which implies $\gamma(f) = \gamma(g)$ since $(I_a * I_p)(g) = (I_a * I_p)(f)$, $(I_p * I_a)(f) = (I_p * I_a)(g)$ and $(I_a * I_p)(f) \neq (I_p * I_a)(f)$. $\diamondsuit$

Claim 3. Let $f, g \in \mathcal{F}$. If either $(I_a * I_p)(f) < (I_p * I_a)(f)$ and $(I_a * I_p)(g) < (I_p * I_a)(g)$, or $(I_a * I_p)(f) > (I_p * I_a)(f)$ and $(I_a * I_p)(g) > (I_p * I_a)(g)$ then $\gamma(f) = \gamma(g)$.

Proof.

Let $f, g \in \mathcal{F}$ be such that either $(I_a * I_p)(f) < (I_p * I_a)(f)$ and $(I_a * I_p)(g) < (I_p * I_a)(g)$, or $(I_a * I_p)(f) > (I_p * I_a)(f)$ and $(I_a * I_p)(g) > (I_p * I_a)(g)$. Let us define h by:

$$h = \frac{(I_p * I_a)(f) - (I_a * I_p)(f)}{(I_p * I_a)(g) - (I_a * I_p)(g)} g + \frac{(I_p * I_a)(g)(I_a * I_p)(f) - (I_p * I_a)(f)(I_a * I_p)(f)}{(I_p * I_a)(g) - (I_a * I_p)(g)}$$

We can easily check that $(I_a * I_p)(h) = (I_a * I_p)(f)$ and $(I_p * I_a)(h) = (I_p * I_a)(f)$, which implies that $(I_a * I_p)(h) \neq (I_p * I_a)(h)$. Claim 1 implies $\gamma(h) = \gamma(g)$. Since either $(I_a * I_p)(f) < (I_p * I_a)(f)$ or $(I_a * I_p)(f) > (I_p * I_a)(f)$, Claim 2 implies $\gamma(f) = \gamma(h)$. Hence, $\gamma(f) = \gamma(g)$.$\diamondsuit$

Claim 3 implies that $\gamma(f)$ only depends on the ordering of $(I_a * I_p)(f)$ and $(I_p * I_a)(f)$, which completes the proof[11]. ■

Proof of Theorem 4.

The "if" part of the Theorem is straightforward. We hence only prove the "only if" part.

Since Axiom (GDOM) is satisfied, so is Axiom (ADOM). It follows from Theorem 3 that there exist $\alpha, \beta \in (0,1)$, such that:

$$I(f) = \begin{cases} \alpha\left(I_p * I_a\right)(f) + (1-\alpha)\left(I_a * I_p\right)(f), & \text{if } \left(I_a * I_p\right)(f) \geq \left(I_p * I_a\right)(f) \\ \beta\left(I_p * I_a\right)(f) + (1-\beta)\left(I_a * I_p\right)(f), & \text{if } \left(I_p * I_a\right)(f) \geq \left(I_a * I_p\right)(f) \end{cases}$$

We want to prove that $\alpha = (1-\beta)$. Let f, g be two matrices in $\mathcal{F}$, different from $\mathbf{0}$, such that $(I_a * I_p)(f) \geq (I_p * I_a)(g) \geq (I_p * I_a)(f) \geq (I_a * I_p)(g)$. We hence get $I(f) = \alpha(I_p * I_a)(f) + (1-\alpha)(I_a * I_p)(f)$ and $I(g) = \beta(I_p * I_a)(g) + (1-\beta)(I_a * I_p)(g)$. Furthermore, Axiom (GDOM) implies $I(f) \geq I(g)$.

Now, let us define h as follows: $h = -f + (I_a * I_p)(f) + (I_p * I_a)(f)$. We can easily check that under Axiom (ADD), $(I_a * I_p)(h) = (I_p * I_a)(f)$ and $(I_p * I_a)(h) = (I_a * I_p)(f)$. Therefore, $(I_p * I_a)(h) > (I_a * I_p)(h)$, which entails $I(h) = \beta(I_p * I_a)(h) + (1-\beta)(I_a * I_p)(h) = \beta(I_a * I_p)(f) + (1-\beta)(I_p * I_a)(f)$. Furthermore, Axiom (GDOM) implies that $I(f) = I(h)$. Hence, $\beta = (1-\alpha)$, which completes the proof[12]. ■

Proof of Theorem 5.

The "if" part of the Theorem is easily checked. We will on prove the "only if" part of the Theorem.

Claim 1. $I \in \mathcal{W}_2 \cap \mathcal{W}_3$ and $I(f) \neq \frac{1}{2}$, $(I_p * I_a)(f) + \frac{1}{2}(I_a * I_p)(f)$ implies that either $(I_p * I_a)(f) \geq (I_a * I_p)(f)$ for all $f \in \mathcal{F}$, or $(I_a * I_p)(f) \geq (I_p * I_a)(f)$ for all $f \in \mathcal{F}$.

Proof.

Let us assume $I \in \mathcal{W}_2 \cap \mathcal{W}_3$ and $I(f) \neq \frac{1}{2}(I_p * I_a)(f) + \frac{1}{2}(I_a * I_p)(f)$. In that case, there exist α and δ in $(0,1) \setminus \left\{\frac{1}{2}\right\}$, such that, for all $f \in \mathcal{F}$,

$$I(f) = \alpha\left(I_p * I_a\right)(f) + (1-\alpha)\left(I_a * I_p\right)(f), \tag{1}$$

and:

$$I(f) = \delta \min\left\{(I_a * I_p)(f), (I_p * I_a)(f)\right\} + (1-\delta)\max\left\{(I_a * I_p)(f), (I_p * I_a)(f)\right\}. \tag{2}$$

Let us assume that there exist f and g in $\mathcal{F}$, such that $(I_a * I_p)(f) > (I_p * I_a)(f)$ and $(I_a * I_p)(g) < (I_p * I_a)(g)$. Using equations (1) and (2), $(I_a * I_p)(f) > (I_p * I_a)(f)$ implies $\alpha = \delta$, whereas $(I_a * I_p)(g) < (I_p * I_a)(g)$ implies $\alpha = (1-\delta)$. But we had assumed that $\alpha \neq \frac{1}{2}$ and $\delta \neq \frac{1}{2}$, which yields a contradiction. ♢

[11]The uniqueness of I_a and I_p, as well as that of α and β, directly follow from Theorem 3.

[12]Uniqueness of I_a and I_p as well as uniqueness of δ directly follow from theorem 3.

Claim 2. If for all $f \in \mathcal{F}$, $(I_a * I_p)(f) \leq (I_p * I_a)(f)$, then I_a is the mathematical expectation or I_p the minimum operator.

Proof.

Let $\tilde{\mathcal{C}}_{I_p} = \left\{q \in \mathcal{C}_{I_p} \,|q_i \geq q_{i+1},\, \forall\, i < n\right\}$ and $\tilde{\mathcal{C}}_{I_a} = \{p \in \mathcal{C}_{I_a} \,|p_\sigma \geq p_{\sigma+1},\, \forall\, \sigma < s\}$. Define: $k_0 = \max\left\{i \,\middle|\, \forall q \in \tilde{\mathcal{C}}_{I_p},\, q_i > 0\right\}$ and $\mathcal{C}^0_{I_p} = \left\{q \in \tilde{\mathcal{C}}_{I_p} \,|\forall\, i > k_0,\, q_i = 0\right\}$. Observe that if $k_0 = 1$, then I_p is the minimum operator.

Clearly, for all $\lambda \in \mathbb{R}^n_+$, such that $\lambda_1 \leq ... \leq \lambda_n$, $\arg\min_{q \in \mathcal{C}_{I_p}} q \cdot \lambda \subset \mathcal{C}^0_{I_p}$. Similarly, for all $\mu \in \mathbb{R}^s_+$, such that $\mu_1 \leq ... \leq \mu_s$, $\arg\min_{p \in \mathcal{C}_{I_a}} p \cdot \mu \subset \tilde{\mathcal{C}}_{I_a}$.

Let us assume $k_0 > 1$ (so, I_p is not the minimum operator). We will now show that this implies that I_a is the mathematical expectation.

Let us define f as follows:

$$\begin{cases} f_{\sigma i} = (i-1)s + \sigma \text{ if } i < k_0 \\ f_{\sigma i} = is - \sigma + 1 \text{ if } i \geq k_0 \end{cases}$$

The matrix f has the following form:

$$f = \begin{pmatrix} 1 & s+1 & \cdots & (k_0-2)s+1 & k_0 s & \cdots & ns \\ 2 & s+2 & \cdots & (k_0-2)s+2 & k_0 s - 1 & \cdots & ns-1 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ s-1 & 2s-1 & \cdots & (k_0-1)s-1 & (k_0-1)s+2 & \cdots & (n-1)s+2 \\ s & 2s & \cdots & (k_0-1)s & (k_0-1)s+1 & \cdots & (n-1)s+1 \end{pmatrix}$$

We can easily check that there exist $q \in \mathcal{C}^0_{I_p}$ and $p \in \tilde{\mathcal{C}}_{I_a}$, such that:

$$I_a(f_{\cdot i}) = (i-1)s + \sum_\sigma \sigma p_\sigma \,\forall i \in K,$$

and

$$I_p(f_{\sigma \cdot}) = s\sum_{i=1}^{k_0} i q_i - s + q_{k_0}(1+s) + \sigma(1 - 2q_{k_0}).$$

Since $I_a(f_{\cdot i}) \leq I_a(f_{\cdot(i+1)})$ for all $i < n$, and $I_p(f_{\sigma \cdot}) \leq I_p(f_{(\sigma+1)\cdot})$ for all $\sigma < s$, this entails that:

$$(I_p * I_a)(f) = \sum_\sigma \sigma p_\sigma + s\sum_{i=1}^{k_0} i q_i - s$$

and:

$$(I_p * I_a)(f) = s\sum_{i=1}^{k_0} i q_i - s + q_{k_0}(1+s) + (1 - 2q_{k_0})\sum_\sigma \sigma p_\sigma.$$

Therefore, $(I_p * I_a)(f) \geq (I_a * I_p)(f)$ if, and only if:

$$2q_{k_0}\sum_\sigma \sigma p_\sigma \geq q_{k_0}(1+s).$$

But, by assumption, $q_{k_0} > 0$. Hence,$(I_p * I_a)(f) \geq (I_a * I_p)(f)$ if, and only if:

$$\sum_\sigma \sigma p_\sigma \geq \frac{1+s}{2}. \tag{3}$$

But $p \in \tilde{\mathcal{C}}_{I_a}$. Which implies that:

$$\sum_\sigma \sigma p_\sigma \geq \frac{1}{s}\sum_\sigma \sigma = \frac{1+s}{2},$$

with an equality if, and only if, $p_\sigma = \frac{1}{s}$ for all σ. Therefore, equation (3) is satisfied if, and only if, $p_\sigma = \frac{1}{s}$ for all σ, i.e., I_a is the mathematical expectation, the desired result.

On the other hand, if $k_0 = 1$, i.e., if I_p is the minimum operator, then clearly, for all $f \in \mathcal{F}$, $(I_p * I_a)(f) \geq (I_a * I_p)$. $\diamondsuit$

Claim 3. If for all $f \in \mathcal{F}$, $(I_a * I_p)(f) \geq (I_p * I_a)(f)$, then I_p is the mathematical expectation or I_a is the minimum operator.

Proof.
By symmetry, the proof is similar to the proof of Claim 2. $\diamondsuit$

By Claims 1 through 3, we can conclude that if $\delta \in (0,1) \setminus \left\{\frac{1}{2}\right\}$ and $I \in \mathcal{W}_2 \cap \mathcal{W}_3$, then I_a or I_p is either the mathematical expectation or the minimum operator.

Finally, let us assume that $\succeq$ is represented by I in $\mathcal{W}_3$ and respects Axiom (GDOM). Clearly, if for all $f \in \mathcal{F}$:

$$I(f) = \frac{1}{2}(I_p * I_a)(f) + \frac{1}{2}(I_a * I_p)(f)$$

then $I \in \mathcal{W}_2$, which completes the proof. $\blacksquare$

Proof of Theorem 6.
The "if" part of the Theorem is straightforward. We hence only prove the "only if" part.

Since $I \in \mathcal{W}$ and I satisfies Axiom (ADD), I_a and I_p are affine.

Let $f \in \mathcal{F}$, such that $(I_p * I_a)(f) \neq (I_a * I_p)(f)$ and consider g defined by: $g_{\sigma i} = \frac{1}{2}I_a(f_{\cdot i}) + \frac{1}{2}I_p(f_{\sigma \cdot})$, for all i in K and all σ in S. We then obtain:

$$\begin{cases} I_a(g_{\cdot i}) = I_a(f_{\cdot i}) + [\frac{1}{2}(I_a * I_p)(f) - \frac{1}{2}I_a(f_{\cdot i})], \forall i \in K \\ I_p(g_{\sigma \cdot}) = I_p(f_{\sigma \cdot}) + [\frac{1}{2}(I_p * I_a)(f) - \frac{1}{2}I_p(f_{\sigma \cdot})], \forall \sigma \in S \end{cases}$$

Now, let us define $u \in \mathbb{R}^n$ by $u_i = \frac{1}{2}(I_a(f_{\cdot i}) - (I_a * I_p)(f))$, for all i in K, and $v \in \mathbb{R}^s$ by $v_\sigma = \frac{1}{2}((I_p * I_a)(f) - I_p(f_{\sigma \cdot}))$. We hence have: $I_a(g) = I_a(f) - u$ and $I_p(g) = I_p(f) + v$

Let $k_1 = \max_\sigma I_p(f_{\sigma \cdot})$, and $k = \max\{k_1, (I_a * I_p)(f)\}$. Clearly, $\mathbf{u}_{\sigma \cdot} + \mathbf{k}$ and $\mathbf{v}_{\cdot i} + \mathbf{k}$ belong to $\mathcal{F}$.

Without loss of generality, we assume that I is normalized with $I(\mathbf{1}) = 1$. Then, we can easily check that, since I_a and I_p are affine, $(I_a * I_p)(\mathbf{u}_{\sigma\cdot} + \mathbf{k}) = (I_p * I_a)(\mathbf{u}_{\sigma\cdot} + \mathbf{k}) = \frac{1}{2}[(I_p * I_a)(f) - (I_a * I_p)(f)] + k$, which implies, since I is a WCI functional, that $I(\mathbf{u}_{\sigma\cdot} + \mathbf{k}) = \frac{1}{2}[(I_p * I_a)(f) - (I_a * I_p)(f)] + k$. Similarly, $(I_a * I_p)(\mathbf{v}_{\cdot i} + \mathbf{k}) = (I_p * I_a)(\mathbf{v}_{\cdot i} + \mathbf{k}) = \frac{1}{2}[(I_p * I_a)(f) - (I_a * I_p)(f)] + k$. Therefore, $I(\mathbf{v}_{\cdot i} + \mathbf{k}) = \frac{1}{2}[(I_p * I_a)(f) - (I_a * I_p)(f)] + k$, from which it follows that $I(\mathbf{u}_{\sigma\cdot} + \mathbf{k}) = I(\mathbf{v}_{\cdot i} + \mathbf{k})$. Therefore, $v \in \mathcal{S}(u)$. Hence, by Axiom (SYM), we have $f \sim g$. By Axiom (ADD), however, we obtain:

$$I(g) = \frac{1}{2}(I_p * I_a)(f) + \frac{1}{2}(I_a * I_p)(f).$$

Hence, for all f such that $(I_a * I_p)(f) \neq (I_p * I_a)(f)$, $\gamma(f) = \frac{1}{2}$. Finally, if $(I_a * I_p)(f) = (I_p * I_a)(f)$, one obviously get $I(f) = \frac{1}{2}(I_a * I_p)(f) + \frac{1}{2}(I_p * I_a)(f)$. ∎

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