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Asymptotics for the L^p-Deviation of the Variance Estimator of a Diffusion

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Abstract

We consider a diffusion process X_t regularized with (small) sampling parameter ε. As in Berzin, León & Ortega (2001),we consider a kernel estimate $\widehat{\alpha}_\varepsilon$ with window $h(\varepsilon)$ of a function α of its variance. In order to exhibit global tests of hypothesis, we derive here central limit theorems for the L^p deviations such as

$$\frac{1}{\sqrt{h}}\left(\frac{h}{\varepsilon}\right)^{\frac{p}{2}}\left(\|\widehat{\alpha}_\varepsilon-\alpha\|_p^p-\mathbb{E}\,\|\widehat{\alpha}_\varepsilon-\alpha\|_p^p\right).$$

Key words. Variance estimator, diffusion process, kernel, L^p-deviation, Central Limit Theorem.

Résumé

Observant un processus de diffusion X_t regularisé en utilisant un (petit) paramètre d'échantillonnage ε, nous considérons, comme Berzin, León & Ortega (2001), un estimateur à noyau $\widehat{\alpha}_\varepsilon$ d'une fonction α de sa variance. En vue d'exhiber des tests d'hypothèses globales sur cette fonction α, nous prouvons ici un théorème de limite centrale pour des déviations L^p de la forme

$$\frac{1}{\sqrt{h}}\left(\frac{h}{\varepsilon}\right)^{\frac{p}{2}}\left(\|\widehat{\alpha}_\varepsilon-\alpha\|_p^p-\mathbb{E}\,\|\widehat{\alpha}_\varepsilon-\alpha\|_p^p\right),$$

où $h(\varepsilon)$ désigne la de fenêtre de l'estimateur à noyau considéré.

Mots Clef. Variance d'une diffusion, noyau de convolution, déviation L^p, théorème de limite centrale.

AMS subject classifications. 60J60, 62M05, 62M02, 60F05, 60F25, 60H05

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1 Introduction and main results

Let $(W_t)_{t\geq 0}$ be a standard Brownian motion. We are given a diffusion process defined by the equation

$$dX_t = \sigma(t)dW_t + b(X_t)dt, \qquad \text{where } \sigma > 0. \tag{1}$$

In this work we consider the estimation of the function $\sigma(t)$ when the observed process is

$$X_t^\varepsilon = \frac{1}{\varepsilon}\int_{-\infty}^{\infty} \varphi\left(\frac{t-u}{\varepsilon}\right) X_u du. \tag{2}$$

More precisely as in [2], we consider a function $G \in L^2(\phi)$ with $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ and continuous and symmetric densities φ and K with support in $[-\frac{1}{2}, \frac{1}{2}]$.

For any $q \geq 1$, define $\|f\|_q = \left(\int_{-\infty}^{\infty} |f(t)|^q \, dt\right)^{\frac{1}{q}}$. We now set

$$\widehat{\alpha}_\varepsilon(t) = \frac{1}{h}\int_{-\infty}^{\infty} K\left(\frac{t-u}{h}\right) G\left(\frac{\sqrt{\varepsilon}}{\|\varphi\|_2}\dot{X}_\varepsilon(u)\right) du, \tag{3}$$

in the previous relation $h = h(\varepsilon) \to 0$ as $\varepsilon \to 0$ is the smoothing parameter: the dependence of h on ε is implicit throughout the paper. Then $\widehat{\alpha}_\varepsilon(t)$ is the non-parametric kernel estimate of the parameter

$$\alpha(t) = \mathbb{E}[G(\sigma(t)Z)], \qquad t \in [0, 1]. \tag{4}$$

where $Z \sim \mathcal{N}(0, 1)$ will denote a standard Normal random variable throughout the paper. Berzin *et alii* [2] quote several interesting special cases:

- if $G(x) = x^2$ then $\alpha(t) = \sigma^2(t)$ (recall that $\mathbb{E}|Z|^2 = 1$),
- if $G(x) = \sqrt{\frac{\pi}{2}}|x|$ then $\alpha(t) = \sigma(t)$ (recall that $\mathbb{E}|Z| = \sqrt{\frac{2}{\pi}}$),
- if $G(x) = \log|x| - 2\gamma$ then $\alpha(t) = \log\sigma(t)$. For this, quote that the constant γ also writes $\gamma = \int_0^\infty \log x \ \phi(x)\, dx = 0.57721566\cdots$.

Using the notion of stable convergence, as in [2], we shall prove that this is enough to consider $b \equiv 0$. In this case our process is a time change Brownian motion.

Define

$$\beta_\varepsilon(t) = \sqrt{h/\varepsilon}(\widehat{\alpha}_\varepsilon(t) - \mathbb{E}\widehat{\alpha}_\varepsilon(t)) \tag{5}$$

a pointwise Central Limit Theorem

$$\beta_\varepsilon(t) \xrightarrow{\mathcal{D}}_{\varepsilon\to 0} \mathcal{N}(0, \Sigma^2(t))$$

is proved for a suitable $\Sigma^2(t)$ explicited in [2]. Alternative estimation techniques and CLTs are proposed in Soulier [11], Genon-Catalot *et alii* [5] and in Brugière [3] under close frames.

Another expression will also be usefull

$$\widehat{\beta}_\varepsilon(t) = \sqrt{h/\varepsilon}(\widehat{\alpha}_\varepsilon(t) - \alpha(t)). \tag{6}$$

If the parameter function is C^2 then the bias of the estimator $\widehat{\alpha}_\varepsilon(t)$ is classically $O(h^2)$ and if one sets $h = \varepsilon^{1/5}$, the centering in the previous convergence may be replaced by $\alpha(t)$ to the price that the Gaussian limit has a non zero mean. The reader is deferred to proposition 1 below for precisions.

In the present paper, our aim is to provide global estimation of the parameter α in L^p for $p \geq 1$. We consider the L^p deviations

$$D_{p,\varepsilon} = \frac{1}{\sqrt{h}}\left(\|\beta_\varepsilon\|_p^p - \mathbb{E}\|\beta_\varepsilon\|_p^p\right), \quad \text{and} \tag{7}$$

$$\mathcal{D}_{p,\varepsilon} = \frac{1}{\sqrt{h}}\left(\|\widehat{\beta}_\varepsilon\|_p^p - \mathbb{E}\|\widehat{\beta}_\varepsilon\|_p^p\right). \tag{8}$$

Such expressions may be used to make global proofs of hypotheses on the diffusion's variance. Such questions have a very special interest in finance. Expression (7) may be used with resampling. Soulier [11] proves a CLT for the case $p = 2$ under wavelet estimation frame. Using a Poissonnization argument Beirlant & Mason [1] obtain analogue results for the difficult case of kernel density and regression estimates based on independent samples.

Let us expand the (even) function $g_t(x) = G(\sigma(t)x)$ in terms of Hermite polynomials

$$g_t(x) = \sum_{n=0}^{\infty} a_{2n}(t)H_{2n}(x), \text{ with } a_{2n}(t) = \frac{1}{2n!}\mathbb{E}G(\sigma(t)Z)\cdot H_{2n}(Z). \tag{9}$$

We set $f \star g$ for the convolution of f and g. For $t \in [0,1]$ and $w \in [-1,1]$, we define

$$\Sigma^2(t) = \|K\|_2^2 \sum_{n=1}^{\infty} a_{2n}^2(t)\ (2n)! \int_{-1}^{1} \left(\frac{\varphi \star \varphi(z)}{\|\varphi\|_2^2} \right)^{2n} dz, \qquad \text{and} \tag{10}$$

$$\Gamma(w) = \frac{K \star K(w)}{\|K\|_2^2} \in [-1,1]. \tag{11}$$

Let (Z_1, Z_2) be a standard $(0, I_2)$ Normal vector, set using notation (10)

$$\Sigma_p^2 = \int_{-1}^{1} \mathrm{Cov}\left(\left| \sqrt{1-\Gamma^2(w)} Z_1 + \Gamma(w) Z_2 \right|^p, |Z_2|^p \right) dw \cdot \int_0^1 \Sigma^{2p}(t)\, dt. \tag{12}$$

Theorem 1 *Assume that the diffusion (1) is such that the function σ is continuous and $\sigma > 0$ over the compact set $[0,1]$. If moreover there exists some $q \geq 4$ such that $I\!E|G(\sigma(t)Z)|^{pq} < \infty$ and $lim_{\varepsilon\to 0} h = \lim_{\varepsilon\to 0} \varepsilon h^{-2(1-1/q)} = 0$. Then*

$$D_{p,\varepsilon} \xrightarrow{\mathcal{D}}_{\varepsilon\to 0} \mathcal{N}(0, \Sigma_p^2).$$

Remarks. Using lemma 5 below proves that the same CLT holds for

$$\widetilde{D}_{p,\varepsilon} = \frac{1}{\sqrt{h}} \left(\|\beta_\varepsilon\|_p^p - \mathrm{I\!E}|Z|^p \int_0^1 (\Sigma(t))^p dt \right)$$

and where $\Sigma^2(t)$ is defined in eqn. (10) is also the limiting variance in [2]'s CLT.

Let $b_{2k} = \frac{1}{2k!} \mathrm{I\!E}|Z|^p H_{2k}(Z)$, then we also may write

$$\Sigma_p^2 = \sum_{k=1}^{\infty} b_{2k}^2\, (2k)! \int \Gamma^{2k}(w)\, dw \int \Sigma^{2p}(t)\, dt.$$

As in Jacod [7], we shall work as previously in [2], the proof of the theorem will use two steps, the first one assumes that $b = 0$, which mean that $X(t)$ is a Brownian with a time change. Hence a first section of the paper will be devoted to recall facts related with this special case.

Proposition 1 *Assume that the even function G is C^2 a.s. and assume that $\sigma > 0$ is a C^2-function. Let $b_\varepsilon(t) = I\!E\widehat{\alpha}_\varepsilon(t) - \alpha(t)$. If moreover $\lim_{\varepsilon\to 0} h = \lim_{\varepsilon\to 0} \frac{\varepsilon}{h} = 0$ then*

$$\lim_{\varepsilon\to 0} h^{-2} \sup_{t\in[0,1]} \left| b_\varepsilon(t) - \frac{h^2}{2} \alpha''(t) \int s^2 K(s) ds \right| = 0.$$

If moreover, $\lim_{\varepsilon\to 0}\frac{\varepsilon^2}{h^3}=0$ *and the functions* G,σ *are* C^3 *the norming factor* h^{-2} *may be replaced by* h^{-3}.

As usual, use of kernels K with higher order yields $b_\varepsilon(t)=c\alpha^{(r)}(t)h^r+o(h^r)$.

We now turn to the asymptotic behaviour of $\mathcal{D}_{p,\varepsilon}$. Still assuming that the functions σ, G are *a.s.* twice differentiable, then the suboptimal window case, $\lim_{\varepsilon\to 0} h^5/\varepsilon=0$ leads to the same result as theorem 1.

$$\mathcal{D}_{p,\varepsilon}\xrightarrow{\mathcal{D}}_{\varepsilon\to 0}\mathcal{N}(0,\Sigma_p^2)\qquad\text{if}\qquad\lim_{\varepsilon\to 0}\frac{h^5}{\varepsilon}=0. \tag{13}$$

We now examine the optimal window case, $h=\lambda\varepsilon^{\frac{1}{5}}$

Theorem 2 *Assume that the function* $\sigma>0$ *is* C^2*, and that* G *is a.s. twice derivable and has a second order bounded derivative we set* $h=\lambda\varepsilon^{\frac{1}{5}}$ *for some constant* $\lambda>0$. *If* $\mathbb{E}|G(\sigma(t)Z)|^{2p}<\infty$ *then*

$$\mathcal{D}_{p,\varepsilon}\xrightarrow{\mathcal{D}}_{\varepsilon\to 0}\mathcal{N}(0,\tau_p^2).$$

where, as in theorem 1, $\tau_p^2=\int\int\Theta(w,t)\,\Sigma^{2p}(t)dw\,dt$ *where, using notations in proposition 1,* $c(t)=\lambda^{\frac{5}{2}}a(t)\int s^2K(s)\,ds$, *and*

$$\Theta(w,t)=Cov\left(\left|\sqrt{1-\Gamma^2(w)}Z_1+\Gamma(w)Z_2+c(t)\right|^p,|Z_2+c(t)|^p\right).$$

Examples. In some special cases of interest, the function G is homogeneous $G(\sigma x)=\sigma^rG(x)$ for $\sigma>0$ hence $\Sigma^2(t)=A\sigma^{2r}(t)$ for a suitable constant $A>0$ only depending on ϕ and on G and, this makes much simpler the expressions of Σ_p^2 and τ_p^2. Examples of this situation $G(x)=\sqrt{\frac{\pi}{2}}|x|$ and $G(x)=x^2$ have already been sketched.

Analogue considerations are valid for the function $G(x)=\log|x|-\gamma$ for which only $a_0(t)=\log\sigma(t)-\gamma$ really depends on t while $a_{2n}(t)=a_{2n}=\frac{1}{2n!}\mathbb{E}\log|Z|H_{2n}(Z)$ for $n>0$, and $\Sigma^2(t)=\Sigma_\varphi^2$ only depends on φ. Quote that Σ_p^2 does not depend on the function $\sigma(\cdot)$; this however does not hold for the companion variance τ_p^2.

The paper is organized as follows, this first section introduces the problem and gives the main results. Section 2 is devoted to a series of technical lemmas usefull in the proof of the main results. The main results are proved in section 3 while the proof of those preliminary lemmas is given in section 4.

2 Collecting some facts in the case $b \equiv 0$

The following simple facts are essentially collected from [2]. Set

$$\dot{\sigma}_\varepsilon^2(t) = \mathrm{Var}\ \dot{X}_\varepsilon(t). \tag{14}$$

Lemma 1 *We have*

$$\begin{aligned} Cov\ (\dot{X}_\varepsilon(s), \dot{X}_\varepsilon(t)) &= \frac{1}{\varepsilon^2}\int \varphi\left(\frac{s-u}{\varepsilon}\right)\varphi\left(\frac{t-u}{\varepsilon}\right)\sigma^2(u)\,du \\ &= \frac{1}{\varepsilon}\int \varphi(x)\varphi\left(x+\frac{t-s}{\varepsilon}\right)\sigma^2(t-\varepsilon x)\,dx \end{aligned}$$

Note that this expression vanishes if $|t-s| > 2\varepsilon$*:*

Then it follows that $\sqrt{\varepsilon}\dot{\sigma}_\varepsilon(t) \to \|\varphi\|_2\sigma(t)$ *as* $\varepsilon \to 0$ *where the previous convergence holds uniformly on* $[0,1]$.

We often work with the following "almost" white noise process which we shall denote for simplicity's sake

$$Z_\varepsilon(t) = \frac{\dot{X}_\varepsilon(t)}{\dot{\sigma}_\varepsilon(t)} \sim \mathcal{N}(0,1) \tag{15}$$

Setting $\rho_\varepsilon(s,t) = \mathrm{Cov}(Z_\varepsilon(s), Z_\varepsilon(t))$, note that the previous lemma implies

$$\rho_\varepsilon(s,t) = \frac{\int \varphi(x)\varphi\left(x+\frac{t-s}{\varepsilon}\right)\sigma^2(t-\varepsilon x)\,dx}{\sqrt{\int \varphi^2(x)\sigma^2(s-\varepsilon x)\,dx\ \int \varphi^2(x)\sigma^2(t-\varepsilon x)\,dx}},$$

this yields

$$\mathbb{E}(\beta_\varepsilon(t))^2 \sim \frac{h}{\varepsilon}\int\int K(u)K(v)\,\mathrm{Cov}(G(\sigma(t)Z_\varepsilon(t-uh)), G(\sigma(t)Z_\varepsilon(t-vh)))dudv$$

The above covariance is a function of t and of $\rho_\varepsilon(t-uh, t-vh)$. Now Mehler formula proves that

$$\begin{aligned} \mathbb{E}(\beta_\varepsilon(t))^2 &\sim \frac{h}{\varepsilon}\int\int K(u)K(v)\sum_{n=1}^{\infty} a_{2n}(t-uh)a_{2n}(t-vh)(2n)! \\ &\times \left(\frac{\int \varphi(x)\varphi(x+\frac{h(v-u)}{\varepsilon})\sigma^2(t-uh-\varepsilon x)dx}{\int \varphi^2(x)\sigma^2(t-uh-\varepsilon x)dx}\right)^{2n} dudv \end{aligned}$$

Finally, the change of variable $z = \frac{h(v-u)}{\varepsilon}$ implies $\mathbb{E}(\beta_\varepsilon(t))^2 \to \Sigma^2(t)$ as $\varepsilon \to 0$ where eqn. (10) defines $\Sigma(t)$. Even if it is asymptotically Gaussian, the process $\beta_\varepsilon(t)$ is not Gaussian and L^p-norms cannot be deduced from Mehler formula. Another way to proceed is used in Giné *et alii* [6]. We also quote that $\beta_\varepsilon(t)$ may be rewritten as the partial sum of 1-dependent random variables.

Lemma 2 *Set* $N = 2\left[\frac{h}{2\varepsilon}\right]$, *then we have* $\beta_\varepsilon(t) = \sum_{k=1}^{N} \zeta_{k,\varepsilon}(t)$ *with*

$$\begin{aligned}\zeta_{k,\varepsilon}(t) &= \int_{-\frac{1}{2}+\frac{k-1}{N}}^{-\frac{1}{2}+\frac{k}{N}} K(u)\left(G\left(\frac{\sqrt{\varepsilon}\dot{\sigma}_\varepsilon(t-uh)}{\|\varphi\|_2}\cdot Z_\varepsilon(t-uh)\right)\right.\\ &- \left.\mathbb{E}G\left(\frac{\sqrt{\varepsilon}\dot{\sigma}_\varepsilon(t-uh)}{\|\varphi\|_2}\cdot Z_\varepsilon(t-uh)\right)\right)du\end{aligned}$$

and the N *random variables* $\zeta_{k,\varepsilon}(t)$ *are 1-dependent for* $k = 1, \ldots, N$.

Since the process $Z_\varepsilon(t)$ is $2\varepsilon-$dependent we obtain this lemma from relation $h/N > \varepsilon$. This also yields

Lemma 3 *Set* $M = \left[\frac{1}{2\varepsilon}\right]$, *then* $\|\beta_\varepsilon\|_p^p = \sum_{\ell=1}^{M} Y_{k,\varepsilon}$ *with*

$$\begin{aligned}Y_{k,\varepsilon} &= \left(\frac{h}{\varepsilon}\right)^{p/2}\int_{(\ell-1)/M}^{\ell/M}\left|\int K(u)\left(G\left(\frac{\sqrt{\varepsilon}\dot{\sigma}_\varepsilon(t-uh)}{\|\varphi\|_2}\cdot Z_\varepsilon(t-uh)\right)\right.\right.\\ &- \left.\left.\mathbb{E}G\left(\frac{\sqrt{\varepsilon}\dot{\sigma}_\varepsilon(t-uh)}{\|\varphi\|_2}\cdot Z_\varepsilon(t-uh)\right)\right)du\right|^p dt\end{aligned}$$

and the M *random variables* $Y_{\ell,\varepsilon}$ *are* $2-$*dependent for* $\ell = 1, \ldots, M$ *if we assume morever that* $h < \varepsilon/2$.

Hence the technique of proof of the main theorem will be based on a Lindeberg Central Limit theorem for $m-$dependent random variables. The two first moments of the above random variable are difficult to calculate directly and thus, in order to avoid this problem we shall proceed as in Giné *et alii* [6]: we use a Gaussian approximation of the previous sums $\beta_\varepsilon(t)$.

The proof of the main theorem will be based on the following serie of lemmatas. They will provide (in particular) the asymptotic L^2 behaviour of $\|\beta_\varepsilon\|_p^p$.

Lemma 4 (approximating expectations) *Let $d \in \mathbb{N}$. Let $x_{1,n}, \ldots, x_{n,n} \in \mathbb{R}^d$ be centered at expectation, $m-$dependent for some integer $m \geq 0$, and such that for some definite $d \times d$ covariance matrix V*

$$\begin{aligned} Var\left(\sum_{k=1}^{n} x_{k,n}\right) &\to_{n\to\infty} V, \text{ and} \\ \sum_{k=1}^{n} \mathbb{E}\|x_{k,n}\|^{3\vee(dpq)} &\to_{n\to\infty} 0. \end{aligned}$$

Write $x_{j,n} = (x_{j,n}^{(\ell)})_{1\leq \ell \leq d}$. Then if $\mathbf{Z} = (Z^{(1)}, \ldots, Z^{(d)}) \sim \mathcal{N}_d(0, V)$, there exists a constant c (only depending on d and on the norm $\|\cdot\|$ on $\mathbb{R}^d$) such that

$$\left| \mathbb{E} \prod_{\ell=1}^{d} \left| \sum_{k=1}^{n} x_{k,n}^{(\ell)} \right|^p - \mathbb{E} \prod_{\ell=1}^{d} \left| Z^{(\ell)} \right|^p \right| \leq c \left(\sum_{k=1}^{n} \mathbb{E}\|x_{k,n}\|^3 \right)^{\delta},$$

where $\delta = 1 - \frac{1}{q}$ if $d = 1$ and $\delta = \frac{1}{4}\left(1 - \frac{pd+d-1}{pqd}\right)$ if $d \geq 2$.

Lemma 5 *Assume that $\lim_{\varepsilon\to 0} h = \lim_{\varepsilon\to 0} \frac{\varepsilon}{h^2} = 0$. Use notation (10), then*

$$\mathbb{E}\|\beta_\varepsilon\|_p^p = \mathbb{E}|Z|^p \int_0^1 (\Sigma(t))^p dt + o\left(\frac{1}{\sqrt{h}}\right), \qquad \text{as } \varepsilon \to 0.$$

In order to provide the asymptotic variance of $D_{p,\varepsilon}$ we make more precise the second order properties of the random process $(\beta_\varepsilon(t))_{t\in[0,1]}$. Set

$$\widetilde{\beta}_\varepsilon(t) = \frac{\beta_\varepsilon(t)}{\sqrt{\operatorname{Var} \beta_\varepsilon(t)}}. \tag{16}$$

To obtain the asymptotic behaviour of $\operatorname{Var} D_{p,\varepsilon}$, we shall need the asymptotic behaviour of $\operatorname{Cov}(\widetilde{\beta}_\varepsilon(s), \widetilde{\beta}_\varepsilon(t))$ easily deduce from the following lemma.

Lemma 6 *Assume that $\lim_{\varepsilon\to 0} h = \lim_{\varepsilon\to 0} \frac{\varepsilon}{h} = 0$, then*

$$\begin{aligned} Cov\,(\beta_\varepsilon(s), \beta_\varepsilon(t)) &\sim \int_{-\frac{1}{2}}^{\frac{1}{2}} K(u) K\left(u + \frac{t-s}{h}\right)\, du \times \\ &\times \int_{-1}^{1} \sum_{n=1}^{\infty} a_{2n}(s) a_{2n}(t) (2n)! \left(\frac{\sigma(t)}{\sigma(s)\|\varphi\|_2^2} \int_{-1}^{1} \varphi(x)\varphi(x+z) dx \right)^{2n} dz. \end{aligned}$$

Mehler formula allows to calculate moments of non linear functionals of a Gaussian process. Hence if the process β_ε was Gaussian we should be able to derive the asymptotic behaviour of $D_{p,\varepsilon}$. But this is not the case. Using Gaussian approximation of β_ε, the following lemma indicates what would be the asymptotic behaviour of $\mathrm{Var}\, D_{p,\varepsilon}$. We thus consider the centered Gaussian process $(B_\varepsilon(t))_{t\in[0,1]}$ such that

$$\mathrm{Cov}\ (B_\varepsilon(s), B_\varepsilon(t)) = \ \mathrm{Cov}\ (\beta_\varepsilon(s), \beta_\varepsilon(t)), \qquad \forall s,t \in [0,1].$$

Lemma 7 *Using notations (10)-(11), we assume that* $\lim_{\varepsilon\to 0} h = \lim_{\varepsilon\to 0} \frac{\varepsilon}{h} = 0$. *Let* $b_{2k} = \frac{1}{2k!} I\!E|Z|^p H_{2k}(Z)$, *then*

$$Var\ \|B_\varepsilon\|_p^p \sim h \sum_{k=1}^{\infty} b_{2k}\,(2k)! \int \Gamma^{2k}(w)\,dw \cdot \int \Sigma^{2p}(t)\,dt.$$

Remark. Let (Z_1, Z_2) be a standard $(0, I_2)$ Normal vector, then the previous expression also writes

$$\mathrm{Var}\|B_\varepsilon\|_p^p \sim h \int \mathrm{Cov}\left(\left|\sqrt{1-\Gamma^2(w)}Z_1 + \Gamma(w)Z_2\right|^p, |Z_2|^p\right) dw \cdot \int \Sigma^{2p}(t)\,dt.$$

3 Proofs of the theorems

3.1 Proof of theorem 1: case $b \equiv 0$

As quoted in lemma 3, $D_{p,\varepsilon}$ is a sum of the $2-$dependent random variables $(Y_{k,\varepsilon} - \mathrm{I\!E} Y_{k,\varepsilon})_{1\le k\le M}$ with $M = M_\varepsilon = \left[\frac{1}{2\varepsilon}\right]$.

Let now $s,t \in [0,1]$ be such that $|s-t| \le 2\varepsilon$, then it is simple to deduce from lemma 2 that the bivariate random variable $(\beta_\varepsilon(s), \beta_\varepsilon(t)) = x_1 + \cdots + x_N$ writes as the sum of $4-$dependent vectors. Thus lemma 4 implies with $d = 2$

$$|\mathrm{I\!E}|\beta_\varepsilon(s)|^p|\beta_\varepsilon(t)|^p - \mathrm{I\!E}|B_\varepsilon(s)|^p|B_\varepsilon(t)|^p| \le \left(\frac{\varepsilon}{h}\right)^{\frac{\delta}{2}}.$$

Apply again this lemma 4 but now with $d = 1$ allows us to substract expectations and this yields finally

$$|\mathrm{Cov}(|\beta_\varepsilon(s)|^p, |\beta_\varepsilon(t)|^p) - \mathrm{Cov}(|B_\varepsilon(s)|^p, |B_\varepsilon(t)|^p)| \le \left(\frac{\varepsilon}{h}\right)^{\frac{\delta}{2}} + o(1),$$

where δ is provided in lemma 4, this is the one for $d = 2$. In order to compute an approximation of Var $D_{p,\varepsilon}$ we first expand it as

$$\text{Var } D_{p,\varepsilon} = \frac{1}{h} \int \int \text{Cov}(|\beta_\varepsilon(s)|^p, |\beta_\varepsilon(t)|^p) \, ds \, dt$$

where $s, t \in [0, 1]$ and with lemma 3, we check that this is enough to assume $|s - t| \leq 2\varepsilon$. Hence the gain provided by the corresponding factor ε proves that the bad value of the rate in lemma 4 is unimportant in the case $d = 2$. Indeed we derive here the bound

$$\left| \text{Var } D_{p,\varepsilon} - \Sigma_p^2 \right| \leq \sqrt{\frac{\varepsilon}{h}} \cdot o(1)$$

which is enough for our purpose.
If $q > 1 + \frac{1}{2p}$, then lemma 7 now yields

$$\text{Var}\,(D_{p,\varepsilon}) \longrightarrow_{\varepsilon \to 0} \Sigma_p^2.$$

The CLT will follow from the Lindeberg condition

$$\eta_\varepsilon = \sum_{k=1}^{M_\varepsilon} \mathbb{E} \left| \frac{1}{\sqrt{h}} (Y_{k,\varepsilon} - \mathbb{E} Y_{k,\varepsilon}) \right|^4 \longrightarrow_{\varepsilon \to 0} 0.$$

Using again lemmas 4, 5 and 6 proves that if $q \geq 4$

$$\mathbb{E}\, |Y_{k,\varepsilon} - \mathbb{E} Y_{k,\varepsilon}|^4 = \mathcal{O}(\varepsilon^4)$$

because it writes also as a the expectation of a quadruple integral on a set with measure M_ε^{-4} and such that the integrand has an expectation uniformly bounded by $2^4 \sup_{t \in [0,1]} \mathbb{E} G^{4p}(\sigma(t) Z)$. This yields

$$\eta_\varepsilon = \mathcal{O}\left(\varepsilon \cdot \left(\frac{\varepsilon}{h} \right)^2 \right).$$

Remark. Set

$$D_{p,\varepsilon,t} = \frac{1}{\sqrt{h}} \int_0^t (|\beta_\varepsilon(s)|^p - \mathbb{E} |\beta_\varepsilon(s)|^p) \, ds. \tag{17}$$

The previous proof provides in fact a Donsker type invariance principle (for $m-$dependent sequences, again). Sketching the expression in theorem 1, we set

$$\Sigma_p^2(t) = \int_{-1}^1 \Theta(w) \, dw \cdot \int_0^t \Sigma^{2p}(s) \, ds.$$

Set $\widetilde{D}_{p,t} = \int_0^t \Sigma_p(s) d\widetilde{W}_s$ for a standard Brownian motion $(\widetilde{W}_t)_{t\in[0,1]}$, then

$$D_{p,\varepsilon,t} \xrightarrow{\mathcal{D}}_{\varepsilon\to 0} \widetilde{D}_{p,t}, \text{ in the space } C([0,1]). \tag{18}$$

3.2 Proof of theorem 1: the general case

Notations. For clarity's sake we add the drift parameter as an index, in the underlying probability law which we now denote $\mathbb{P}^{(b)}$ and expectations $\mathbb{E}^{(b)}$. By another hand, the expression relative to the Brownian motion with a time change (*i.e.* $b \equiv 0$) now write respectively as $E_\varepsilon^{(0)}(t)$, and $\mathbb{E}^{(0)}$.

An essential lemma links the expectations relative to $\mathbb{E}^{(b)}$ and $\mathbb{E}^{(0)}$.

Lemma 8 (Girsanov formula, e.g. in [7]) *Let $H : \mathbb{R} \to \mathbb{R}$ be continuous and bounded then:*

$$\mathbb{E}^{(b)} H(D_{p,\varepsilon}) = \mathbb{E}^{(0)} \left\{ H(D_{p,\varepsilon}) \exp\left(\int_0^1 b(X_s) dX_s - \frac{1}{2}\int_0^1 b^2(X_s)\sigma^2(s) ds \right) \right\}$$

An independence argument called stable convergence is also developped in [7]. It will conclude to the convergence in distribution of $D_{p,\varepsilon}$ under the general law $\mathbb{P}^{(b)}$ with the help of the Cameron-Martin formula which states that

$$\mathbb{E}^{(0)} \exp\left(\int_0^1 b(X_s) dX_s - \frac{1}{2}\int_0^1 b^2(X_s)\sigma^2(s) ds \right) = 1.$$

We are thus aimed to prove that the couple $\Big((X_t)_{t\in[0,1]}, D_{p,\varepsilon}\Big)$ converges in $C([0,1]) \times \mathbb{R}$ (under the distribution $\mathbb{P}^{(0)}$) to $\Big((X)_{t\in[0,1]}, \Sigma_p Z\Big)$ where the Brownian motion with a time change $(X)_{t\in[0,1]}$ is independent of the standard normal Z.

From now, we only work under the probability distribution $\mathbb{P}^{(0)}$. Thus the previous asymptotic independence hold if the bivariate process $(E_{\varepsilon,t})_{t\in[0,1]} \equiv (X_t, D_{p,\varepsilon,t})_{t\in[0,1]}$ converges to a process $(E_t)_{t\in[0,1]} \equiv (X_t, D_{p,t})_{t\in[0,1]}$ as $\varepsilon \to 0$ such that $(X_t)_{t\in[0,1]}$ is independent of $D_{p,1}$ (we shall prove it for $(D_{p,t})_{t\in[0,1]}$).

As the family of distributions $(D_{p,\varepsilon,t})_{t\in[0,1]}$ converges under the probability distribution $\mathbb{P}^{(0)}$ as $\varepsilon \to 0$, this implies its tightness in $C([0,1])$ hence the bivariate process $(E_{\varepsilon,t})_{t\in[0,1]}$ is also tight in $C([0,1], \mathbb{R}^2)$. Consider now any limit point $(E_t)_{t\in[0,1]} \equiv (X_t, D_{p,t})_{t\in[0,1]}$ (in distribution) of this family, as $\varepsilon \to 0$. Since $(D_{p,\varepsilon,t} - D_{p,\varepsilon,s}, X_t - X_s)$ is independent (always under $\mathbb{P}^{(0)}$)

from $D_{p,\varepsilon,t'} - D_{p,\varepsilon,s'}$ if $s \le t \le s' \le t'$ satisfy $s' - t' > 2\varepsilon$ and it is also independent of $X_{t'} - X_{s'}$ because the intervall $[s,t]$ and $[s',t']$ do not overlap. This implies that the process $(E_t)_{t\in[0,1]}$ has independent increments. This process is also a second order process because each of its coordinate have this property. Hence $(E_t)_{t\in[0,1]}$ is a Gaussian process.

Independence of $(E_t)_{t\in[0,1]}$ coordinates now relies on their orthogonality. The only point we need to prove is thus that under the probability distribution $\mathbb{P}^{(0)}$

$$\mathrm{Cov}(X_s, D_{p,\varepsilon,t}) \longrightarrow_{\varepsilon\to 0} 0, \qquad \forall s,t \in [0,1].$$

Consider a Gaussian process $(X_t, B_{\varepsilon,t})_{t\in[0,1]}$ with the same covariance as $(X_t, \beta_{\varepsilon,t})_{t\in[0,1]}$. Note that

$$\mathrm{Cov}\left(X_s, \int_0^t |\beta_{\varepsilon,u}|^p du\right) = \int_0^t \mathrm{Cov}\left(X_s, |\beta_{\varepsilon,u}|^p\right) du. \tag{19}$$

Using lemma 4 with $d=1$, we prove that the previous integral has the same asymptotic behaviour as $\int_0^t \mathrm{Cov}\left(X_s, |B_{\varepsilon,u}|^p\right) du$. Indeed we obtain

$$\left|\mathbb{E}X_s|\beta_{\varepsilon,u}|^p - \mathbb{E}X_s|B_{\varepsilon,u}|^p\right| \le \left(\frac{\varepsilon}{h}\right)^{\{1-1/q\}/2},$$

since, first conditioning w.r.t. X_s makes that lemma 4 applies. In order to proceed we first write $A_\varepsilon(s,u) = \mathrm{Cov}\left(X_s, |\beta_{\varepsilon,u}|^p\right) = 0$ if $u > s+\varepsilon$. Now we deduce that $\int_{s-2\varepsilon}^{s+\varepsilon} A_\varepsilon(s,u)\,du = O\left(\frac{\varepsilon}{\sqrt{h}}\right)$ from the relation $\sup_{t\in[0,1]} \mathbb{E}|\beta_\varepsilon(t)|^{2p} < \infty$. We thus only need to consider $\int_0^{s-2\varepsilon} A_\varepsilon(s,u)\,du$.

Now conditioning distributions keeps Gaussianity for random vectors. Hence, integration w.r.t. X_s's distribution will conclude the above inequality. For this, we still need to check that the conditionned r.v.'s are also $m-$dependent. Write $A_\varepsilon(s,u) = \int x\mathbb{P}_{X_s}(dx)\mathbb{E}(|\beta_\varepsilon(u)|^p/X_s = x)$. Recall that $\beta_\varepsilon(u)$ is a non-linear function of such Z_{u-vh} defined in (15) with $u - vh \le s-\varepsilon$. Conditionning such Z_{u-vh} w.r.t. $X_s = x$ allows to write $Z_{u-vh} = \sqrt{\varepsilon}\nu_{u,v,\varepsilon}x + H_{u,v,\varepsilon}$ for some uniformly bounded and determinist $\nu_{u,v,\varepsilon}$ and some Gaussian $H_{u,v,\varepsilon}$ because $\mathrm{Cov}(X_s, Z_{u-vh}) = O(\sqrt{\varepsilon})$. Moreover the resulting variables are still $m-$dependent. Finally integration of the bound in lemma 4 w.r.t. $\mathbb{P}_{X_s}(dx)$ is legitimated by the bound $(x\vee 1)^r(\varepsilon/h)^{1/2-1/2q}$ for a suitable $r>0$.

The RHS in eqn. (19) vanishes for any $u \in [0,1]$ because the function $x \mapsto |x|^p$ is even while the function $x \mapsto x$ is odd. Now Mehler formula proves that this expression equals 0 which concludes the proof of theorem 1.

3.3 Proof of proposition 1

Write

$$b_\varepsilon(t) = \int K(s)\mathbb{E}G\left(\frac{\sqrt{\varepsilon}\dot{\sigma}(t-hs)}{\|\varphi\|_2}Z\right)\,ds - \alpha(t).$$

Using lemma 1, setting $\theta = \sigma^2$, we obtain the following uniform estimates

$$\left(\frac{\sqrt{\varepsilon}\dot{\sigma}(t-hs)}{\|\varphi\|_2}\right)^2 = \theta(t) - sh\theta'(t) + \frac{1}{2}s^2h^2\theta''(t) + o(h^2).$$

Consider the function $g(x) = G(\sqrt{|x|})$, thus g is also *a.s.* twice differentiable and

$$b_\varepsilon(t) = \int K(s)\mathbb{E}g\left(\left(\frac{\sqrt{\varepsilon}\dot{\sigma}(t-hs)}{\|\varphi\|_2}\right)^2 Z^2\right)\,ds - \mathbb{E}g(\theta(t)Z^2).$$

Use of Taylor formula yields

$$\begin{aligned} b_\varepsilon(t) &= \mathbb{E}\int K(s)\left((-sh\theta'(t) + \frac{1}{2}s^2h^2\theta''(t))Z^2g'\left(\theta(t)Z^2\right)\right. \\ &+ \left.\frac{1}{2}s^2h^2\theta'^2(t)Z^4g''\left(\theta(t)Z^2\right)\right)\,ds + o(h^2). \end{aligned}$$

Using symmetries yields with the relation $g(u) = G(u^2)$,

$$b_\varepsilon(t) = \frac{h^2}{2}\int s^2K(s)\,ds\cdot\mathbb{E}\left(\sigma''(t)ZG'(\sigma(t)Z) + \sigma'^2(t)Z^2G''(\sigma(t)Z)\right) + o(h^2).$$

The remark concerning the C^3 case follows from carefull statements of the above relation with the bound $\varepsilon^2 = o(h^3)$.

3.4 Proof of theorem 2

As previously, we make use of the stable convergence argument in order to deal only with the simpler case $b \equiv 0$. **We assume below that $b \equiv 0$.** Write $\widehat{\beta}_\varepsilon(t) = \beta_\varepsilon(t) + c_\varepsilon(t)$, with

$$c_\varepsilon(t) = \sqrt{\frac{h^5}{\varepsilon}}b_\varepsilon(t) = \sqrt{\frac{h^5}{\varepsilon}}\left(a(t)\int s^2K(s)\,ds + o(1)\right) \qquad (20)$$

We thus write

$$\mathcal{D}_{p,\varepsilon} = \frac{1}{\sqrt{h}} \int_0^1 \left(|\beta_\varepsilon(t) + c_\varepsilon(t)|^p - \mathbb{E}\, |\beta_\varepsilon(t) + c_\varepsilon(t)|^p \right) \, dt$$

Proof of relation (13). Using the bound

$$\Big| |\beta + c|^p - |\beta|^p \Big| \le p|c| \left(|\beta|^{p-1} + |c|^{p-1} \right),$$

we write the following integral (still with $|s-t| \le 2\varepsilon$)

$$\text{Var } (\mathcal{D}_{p,\varepsilon} - D_{p,\varepsilon}) \le \frac{1}{h} \int \int \mathbb{E} |(|\widehat{\beta}_\varepsilon(s)|^p - |\beta_\varepsilon(s)|^p)(|\widehat{\beta}_\varepsilon(t)|^p - |\beta_\varepsilon(t)|^p)| ds\, dt.$$

Assume $\lim_{\varepsilon \to 0} h^5/\varepsilon = 0$ we thus obtain $\lim_{\varepsilon \to 0} \text{Var } (\mathcal{D}_{p,\varepsilon} - D_{p,\varepsilon}) = 0$, hence (13), from the facts that $\sup_{s \in [0,1]} \|\beta_\varepsilon(s)\|_{2p-1} \le \sup_{s \in [0,1]} \|\beta_\varepsilon(s)\|_{2p} < \infty$ and $\varepsilon/h \to 0$.

Proof of theorem 2. Replacing β by $\beta + c$ and c by $c_\varepsilon - c$, we prove as above that

$$\lim_{\varepsilon \to 0} \text{Var } \left(\mathcal{D}_{p,\varepsilon} - \widehat{\mathcal{D}}_{p,\varepsilon} \right) = 0,$$

where

$$\widehat{\mathcal{D}}_{p,\varepsilon} = \frac{1}{\sqrt{h}} \int_0^1 \left(|\beta_\varepsilon(t) + c(t)|^p - \mathbb{E} |\beta_\varepsilon(t) + c(t)|^p \right) \, dt.$$

The proof of theorem 2 now follows the same lines as that of theorem 1 up to very simple changes in lemma 4. This lemma was indeed dedicated to the approximation of $\mathbb{E} f(x_1 + \cdots + x_n)$ for $m-$dependent vector valued sequences and for the special function $f(x_1, \ldots, x_d) = \prod_{\ell=1}^d |x_\ell|^p$, very small changes entail the same result with $f(x_1, \ldots, x_d) = \prod_{\ell=1}^d |x_\ell + c_\ell|^p$ for fixed real numbers $c_1, \ldots, c_d$. Indeed, one may easily rewrite a version of this lemma for which the measurable function f only satisfies $|f(x_1, \ldots, x_d)| \le \prod_{\ell=1}^d |x_\ell|^p \vee 1$.

4 Proofs of the lemmas in section 2

4.1 Proof of lemma 4

The proofs are distinct for $d = 1$ and $d \ge 2$.

Case $m = 1$**.** Shergin ([9], theorem 1) proves that

$$\Delta_n = \sup_{x \in \mathbb{R}} \left| \mathbb{P}\left(\sum_{k=1}^{n} x_{k,n} \le x \right) - \mathbb{P}(Z \le x) \right| \le c \sum_{k=1}^{n} \mathbb{E} \|x_{k,n}\|^3.$$

Recall that the following relation holds for each random variable in L^p

$$\mathbb{E}|X|^p = p \int_0^\infty x^{p-1} \mathbb{P}(|X| > x) dx,$$

the difference of expectations to approximate is thus an integral over $\mathbb{R} = (-\infty, \infty)$. Divide it for $|x| \le M \equiv \Delta_n^{-\frac{1}{pq}}$ and $|x| > M$. Rosenthal inequality [8] up to order pq (this also holds with $m-$dependent sequences since sums may be rewritten as m sum of independent variables) and Markov inequality provide a bound for the the second term while the first one is bounded by using the previous result in [9].

Case $m \ge 2$**.** In order to handle the same technique as above, we need to develop a bound analogue to that in [9].

Lemma 9 *Assume that the assumptions in lemma 4, then*

$$\Delta_n = \sup_{x \in \mathbb{R}^+} \left| \mathbb{P}\left(\left\| \sum_{k=1}^{n} x_{k,n} \right\| \le x \right) - \mathbb{P}(\|\mathbf{Z}\| \le x) \right| \le c \left(\sum_{k=1}^{n} \mathbb{E} \|x_{k,n}\|^3 \right)^{\frac{1}{4}}$$

Notation. For simplicity, set

$$\Delta_n(x) = \left| \mathbb{P}\left(\left\| \sum_{k=1}^{n} x_{k,n} \right\| \le x \right) - \mathbb{P}(\|\mathbf{Z}\| \le x) \right|$$

The proof of lemma 4 now follows the same lines as for $d = 1$ up to the following expressions

$$\begin{aligned} \mathbb{E}|X_1 \cdots X_d|^p &= p^d \int_0^\infty \cdots \int_0^\infty |x_1 \cdots x_d|^{p-1} \\ &\times \ (1 - \mathbb{P}(X_1 \le x_1, \ldots, X_d \le x_d)) \, dx_1 \cdots dx_d. \end{aligned}$$

Using for example $\|(x_1, \ldots, x_d)\| = \max\{|x_1|, \ldots, |x_d|\}$ implies that the difference of product moments to bound is bounded above by

$$c_p \int_0^\infty x^{pd+d-1} \Delta_n(x) dx.$$

for a constant $c_p > 0$ only depending on p. Still using the same tricks as above yields the result, but now the truncation is at level,

$$M^{-4pqd} = \sum_{k=1}^{n} \mathbb{E}\|x_{k,n}\|^3.$$

4.2 Proof of lemma 9

The proof will use the following lemma which is an easy extension of [10] to a vector valued case.

Lemma 10 (Lindeberg-Rio for m-dependent sequences) *Let $d \in \mathbb{N}$. Let $x_1, \dots, x_n \in \mathbb{R}^d$ be centered at expectation, $m-$dependent and such that $\mathbb{E}\|x_k\|^3 < \infty$ for $k = 1, \dots, n$. Then there exists an independent succession $y_1, \dots, y_n$ of centered $d-$dimensional random vectors with the following property. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a C^3-function with bounded partial derivatives of order 3 (write $\|f'''\|_\infty = \sup_{\{s, \|h_i\| \le 1; i=1,2,3\}} \|f'''(s)(h_1, h_2, h_3)\|$), then if we consider*

$$\Delta_n(f) = \mathbb{E}(f(x_1 + \cdots + x_n) - f(x_1 + \cdots + x_n)),$$

there exists a constant $c > 0$ such that

$$|\Delta_n(f)| \le c\|f'''\|_\infty \sum_{k=1}^{n} \mathbb{E}\|x_k\|^3$$

Remarks. In view of the theorem relative to the equivalence of the norms in the $d-$dimensional space we may choose any norm on $\mathbb{R}^d$ and the constant c only depends on this norm and on m.

A simple use of Taylor formula at the origin and with order 3 proves that expression $\Delta_n(f)$ is well defined.

With lemma 10 we consider a C^3-function $g_{\delta,u}$ such that $g_{\delta,u}(x) \in [0, 1]$ for each $x \in \mathbb{R}^d$ and $g_{\delta,u}(x) = 1$ if $\|x\| \le u$, $g_{\delta,u}(x) = 0$ if $\|x\| \ge u + \delta$. This is possible to construct such functions satisfying moreover $\|g'''_{\delta,u}\|_\infty \le c\delta^{-3}$. Let now $\delta^4 = \sum_{k=1}^{n} \mathbb{E}\|x_k\|^3$, then the result follows in a standard way (see e.g. [4]).

4.3 Proof of lemma 10

Notations. The second derivative of f at point s is a (symmetric) bilinear form on $\mathbb{R}^d$. It will also be considered as a (symmetric) $d \times d$ matrix and we shall denote

$$f''(s) \bullet v = \sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j} f(s) \cdot v_{i,j} \text{ if } v = (v_{i,j})_{1\le i,j\le d}.$$

For simplicity we shall handle only the case $m = 1$. We construct independent Gaussian random variables $y_1, \ldots, y_n$ independent of $(x_1, \ldots, x_n)$ and such that $y_k \sim \mathcal{N}(0, v_k)$ with $v_k = \mathbb{E} x_k^t x_k + \mathbb{E} x_{k-1}^t x_k + \mathbb{E} x_k^t x_{k-1}$ for $k = 1, \ldots, n$, where we set $x_0 = 0$.

Remark. In order to complete the proof in the general $m-$dependent case we should have defined

$$v_k = \mathbb{E} x_k^t x_k + \sum_{\ell=1}^{m} \left(\mathbb{E} x_{k-\ell}^t x_k + \mathbb{E} x_k^t x_{k-\ell} \right)$$

for $k = 1, \ldots, n$, where $x_0 = \cdots = x_{1-m} = 0$.

Set $s_k = x_1 + \cdots + x_k$, $t_k = y_{k+1} + \cdots + y_n$ if $k = 0, \ldots, n$, with $s_0 = 0$, $t_n = 0$.

As in ([10], definition 3) we decompose

$$\begin{aligned} \Delta_n(f) &= \mathbb{E}(f(s_n) - f(t_0)) = \sum_{k=1}^{n} \left(\Delta_{1,k}(f_k) - \Delta_{2,k}(f_k) \right), \quad \text{with} \\ \Delta_{1,k}(g) &= \mathbb{E} g(s_k) - g(s_{k-1}) - \frac{1}{2} g''(s_{k-1}) \bullet v_k, \text{ and} \\ \Delta_{2,k}(g) &= \mathbb{E} g(s_{k-1} + y_k) - g(s_{k-1}) - \frac{1}{2} g''(s_{k-1}) \bullet v_k, \text{ and where} \\ f_k(x) &= \mathbb{E} f(x + t_k), \text{ hence } \|f_k'''\|_\infty \le \|f'''\|_\infty. \end{aligned}$$

In the above display $g : \mathbb{R}^d \to \mathbb{R}$ denotes any C^3-function with third order bounded partial derivatives. The bound

$$|\Delta_{2,k}(g)| \le c \|g'''\|_\infty \left(\mathbb{E}\|x_k\|^3 + \|x_{k-1}\|^3 \right) \tag{21}$$

follows from Taylor formula

$$\|g(s + y) - g(s) - g'(s)(y) - \frac{1}{2} g''(s)(y, y)\| \le \frac{1}{6} \|g'''\|_\infty \mathbb{E}\|y\|^3$$

applied with $s = s_{k-1}$ and $y = y_k$ and the independence properties of $y_1, \ldots, y_n$, for a suitable constant c. To convince himself, the reader may restate the formula

$$\mathbb{E} g''(s_{k-1})(y_k, y_k) = \mathbb{E} g''(s_{k-1}) \bullet v_k.$$

The terms $\Delta_{1,k}(g)$ are more delicate to expand. Using again the previous Taylor expansion (now $y = x_k$) we see that, up to a term bounded as in eqn (21), we only need to consider the expectation of

$$g'(s_{k-1})(x_k) + \frac{1}{2}g''(s_{k-1})(x_k, x_k) - \frac{1}{2}g''(s_{k-1}) \bullet v_k = \delta_1 + \delta_2$$

with

$$\begin{aligned} \delta_1 &= g'(s_{k-1})(x_k) - \frac{1}{2}g''(s_{k-1}) \bullet \left(\mathbb{E}x_{k-1}^t x_k + \mathbb{E}x_k^t x_{k-1}\right), \\ \delta_2 &= g''(s_{k-1})(x_k, x_k) - \frac{1}{2}g''(s_{k-1}) \bullet \mathbb{E}x_k^t x_k. \end{aligned}$$

Rewrite

$$\begin{aligned} \delta_2 &= \frac{1}{2}(g''(s_{k-1}) - g''(s_{k-2}))(x_k, x_k) \\ &+ \frac{1}{2}g''(s_{k-2})(x_k, x_k) - \frac{1}{2}g''(s_{k-1}) \bullet \mathbb{E}x_k^t x_k. \end{aligned}$$

Using a first order Taylor expansion yields as before with independence of x_k and s_{k-2}

$$\begin{aligned} \mathbb{E}\delta_2 &= \frac{1}{2}\left(\mathbb{E}g''(s_{k-2})(x_k, x_k) - g''(s_{k-1}) \bullet \mathbb{E}x_k^t x_k\right) \\ &= \frac{1}{2}\left(\mathbb{E}g''(s_{k-2}) - g''(s_{k-1})\right) \bullet \mathbb{E}x_k^t x_k. \end{aligned}$$

The mean value theorem provides now the expected bound for $\mathbb{E}\delta_2$, analogous to that in eqn. (21).

The other term considered writes

$$\delta_1 = g''(s_{k-2})(x_{k-1}, x_k) - \frac{1}{2}g''(s_{k-1}) \bullet \left(\mathbb{E}x_{k-1}^t x_k + \mathbb{E}x_k^t x_{k-1}\right) + R$$

where $|R| \leq c\|g'''\|\mathbb{E}\|x_k\|\|x_{k-1}\|^2$ is bounded as in the above eqn. (21). Hence, using again the mean value theorem implies lemma 3.

4.4 Proof of lemma 5

Set $d = 1$. Fix $t \in [0, 1]$. We apply the approximation lemma 4 to the random variables $x_k = \zeta_{k,\varepsilon}(t)$ for $1 \leq k \leq N$; for simplicity also set $x_0 = x_{N+1} = 0$.

Then setting $y_1, \ldots, y_N$, a sequence of independent and centered random variables such that, $\mathbb{E}y_k^2 = \mathbb{E}x_{k-1}x_k + \mathbb{E}x_k^2 + \mathbb{E}x_k x_{k+1}$ we deduce that, for a suitable constant $c > 0$,

$$|\mathbb{E}|\beta_\varepsilon(t)|^p - \mathbb{E}\,|B_\varepsilon(t)|^p| \leq c \left(\frac{1}{h} \cdot \frac{\varepsilon}{h}\right)^{\frac{1}{2}}. \tag{22}$$

And we also deduce from (10) that

$$\frac{1}{\sqrt{h}} \left(\mathbb{E}\|\beta_\varepsilon\|_p^p - \mathbb{E}|Z|^p \int_0^1 (\Sigma(t))^p dt\right) \rightarrow_{\varepsilon \to 0} 0.$$

4.5 Proof of lemma 6

Using lemma 1 with Mehler formula yields

$$\text{Cov}\ (\beta_\varepsilon(s), \beta_\varepsilon(t)) \sim \frac{h}{\varepsilon} \int \int K(u)K(v) \sum_{n=1}^{\infty} a_{2n}(s) a_{2n}(t)(2n)!\, A_\varepsilon^{2n}(s,t,u,v)\, du\, dv$$

with

$$A_\varepsilon(s,t,u,v) = \frac{1}{\Sigma(s-uh)\Sigma(t-vh)} \int \varphi(x)\varphi\left(x + \frac{t-s}{\varepsilon} + h\frac{u-v}{\varepsilon}\right) \sigma^2(t-vh)\, dx.$$

The change of variable $v \mapsto z = \dfrac{t-s}{\varepsilon} + h\dfrac{u-v}{\varepsilon}$ yields the result if one make use of Lebesgue dominated theorem (the corresponding integrals are uniformly convergent).

4.6 Proof of lemma 7

Using again Mehler formula, we get

$$\text{Var}\ \|B_\varepsilon\|_p^p = \int \int \sum_{k=1}^{\infty} b_{2k}\,(2k)!\, \text{Cov}^{2k}(B_\varepsilon(s), B_\varepsilon(s)) \Sigma^p(t) \Sigma^p(s)\, ds\, dt.$$

Now the change of variable $s \mapsto w = \dfrac{t-s}{h}$ and a systematic use of Lebesgue convergence theorem yield the result.

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