

Econometrics of Share Auctions*

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Abstract

The purpose of this paper is to propose structural econometric methods for the empirical study of Wilson's (1979) share auction model. This is a common value model in which a single and perfectly divisible good is sold to a group of symmetric and risk-neutral buyers. The parameters in the distribution function of the value of the good and the signals received by the buyers are estimated using a two-step estimation procedure. The methods are applied to French Treasury securities auctions held in 1995. A counterfactual comparison shows that the Treasury's revenue in the discriminatory share auction (the mechanism adopted by the French Treasury) is 5% higher than in the uniform share auction.

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1 Introduction

Over the past forty years the theoretical literature on auctions has considerably expanded (see Klemperer, 1999, for a recent survey). Auctions are relatively simple and well-defined economic environments, and the theoretical models often lead to clear-cut predictions. For this reason and because auction data are readily available, the theory has triggered a vast amount of empirical articles (see Hendricks and Paarsch, 1995, for a survey on empirical work). Two types of approaches can be distinguished in the empirical literature. There is first of all the so-called reduced form approach. In reduced form econometrics little or no restrictions are imposed on the data generating process. The objective is typically to provide a descriptive analysis of the auction data, or to test certain predictions of the theory (such as the well-known Revenue Equivalence theorem). The second approach is known as the structural econometric form approach. Structural econometrics consists in methods to estimate the parameters of a particular auction model incorporating all restrictions and constraints of the theory. This approach is of particular interest when one desires to measure the effects of a change in auction design on the behavior of bidders and the revenue of sellers.

Until now the structural econometric literature has focused its attention on the standard first-price auction of a single indivisible good.¹ Various structural estimation techniques have been developed for this auction model under different bidding environments. Donald and Paarsch (1993, 1996), Elyakime, Laffont, Loisel and Vuong (1994), Laffont, Ossard and Vuong (1995), and Guerre, Perrigne and Vuong (2000) consider the Independent Private Value (IPV) paradigm so that the structural element to be estimated is the distribution of private values of the buyers. Li, Perrigne and Vuong (1999) consider an affiliated private value framework so that their objective is to estimate the joint distribution of private values. Finally, Hendricks, Pinkse and Porter (1999) and Hong and Shum (1999) consider the common value paradigm and propose methods to estimate the joint distribution function of the value of the good and the signals received by the buyers.

The purpose of this paper is to propose structural econometric methods for the empirical study of Wilson's (1979) share auction model. This is a common value model in which a single and perfectly *divisible* good is sold to a group of symmetric and risk-neutral buyers. As in the standard common value auction model for an indivisible good, the value of the object is unknown at the time of bidding, and prior to the auction the buyers independently receive signals that are informative about the value. Unlike the standard model, each potential buyer not only submits a price but also a share (or fraction) he requests at that price. Each bidder can submit as many price/share combinations as he wants, thereby constituting a demand curve for the good. By aggregating the individual demand curves, the auctioneer can determine the equilibrium price that clears the market. Given the announced (prior to the auction) allocation mechanism and pricing rule, the shares are then awarded to the winning bidders, who make their payments to the seller. Wilson restricts his analysis primarily to two auction formats: the uniform price auction and the discriminatory auction. In both auction formats the allocation device is the same, and consists in awarding

¹An exception is the paper by Donald, Paarsch and Robert (1997), who consider the estimation of sequential ascending-price auctions with multi-unit demand.

to each bidder the fraction of the good he requested at the equilibrium price. The payment rule, however, differs in the two auction mechanisms. Roughly speaking, in the uniform auction, all winners (bidders with positive demand at the equilibrium price) pay the equilibrium price, whereas in the discriminatory auction, for each marginal share they receive, winners pay the price at which the bid was submitted.

Although basically our estimation procedure can be applied to both the discriminatory and uniform share auction model, we only present the methods for the former auction format. The structural elements of interest are the marginal distribution function of the value of the good, and the conditional distribution function of the signal given the value. We specify the two distribution functions parametrically so that the goal is to estimate a vector of unknown parameters. This is done via a two-step procedure. In the first step we nonparametrically estimate the distribution function of the demand for the good using the observed price/share bids. This first step enables us to estimate the inverse demand function for the good. In the second step, the estimated inverse demand function is inserted into the Euler condition resulting from the bidder's maximization program. The Euler condition can be seen as a set of moment restrictions that depend on the unknown parameters of interest. Minimization of the empirical counterpart of the moment restrictions then leads to our estimator.

Our method of statistical inference only relies on the Euler condition implied by the bidder's maximization problem in the discriminatory auction. Therefore, although it must be assumed that there exists an equilibrium strategy and that all bidders behave according to this strategy, the explicit form of the equilibrium does not need to be known.² The fact that only the first order condition needs to be known is a particularly attractive feature of our estimation method since recent work has shown that, unlike the uniform share auction, it appears difficult to find explicit optimal strategies in the case of the discriminatory share auction (Viswanathan, Wang and Witelski (2000) and Hortaçsu (2001) have obtained explicit solutions for the very specific two-bidder case only).

Our estimator belongs to the class of semiparametric two-step estimators studied by Newey and McFadden (1994). Drawing on their estimation theory for this class of models, we show that our estimator is consistent and asymptotically normally distributed, and we indicate how the asymptotic variance matrix can be consistently estimated.

The methods are applied to data from Treasury securities auctions. Treasury securities auctions are often cited as a good example of a share auction (see for example the recent survey on Treasury auction theory by Das and Sundaram, 1996). Treasury securities auctions are auctions in which huge quantities of strictly identical goods—i.e. securities—are sold to a group of buyers, generally large investment institutions. Since these institutions typically require different amounts of securities at different prices, the Treasury allows them to submit price/quantity combinations, that is a demand curve for the securities. Normalizing the total amount of securities offered to one, and dividing the quantity bids by the total volume, a Treasury auction is indeed a good approximation of

²This is an important difference with the first-price auction papers cited above. The econometric methods developed in those papers all rely on the explicit solution of the optimal bidding function in the first-price auction model. The estimation procedures for the general class of game theoretic models proposed by Florens, Protopopescu, and Richard (2001), also necessitate an explicit solution of the equilibrium strategy.

a share auction.

Although Wilson did not mention Treasury auctions in his article (may be because at that time the practice of selling securities at auction was much less widespread than nowadays, see Bartolini and Cottarelli, 1997), his model and its underlying assumptions are often regarded as well adapted to the context of Treasury auctions. For instance, in their discussion about the theoretical literature on divisible-good auctions, Back and Zender (1993) refer to Wilson's share auction model as the most relevant model for Treasury securities. Bikhchandani and Huang (1993) state that in Treasury securities auctions the common value assumption is appropriate because the value for each bidder is a common and unknown resale price. Each investor is presumed to have some private information about the resale price of the security (coming from forecasts on interest movements, and anticipations of future demand for the security). Finally, Das and Sundaram (1996) argue that bidders in Treasury securities auctions can be considered as symmetric and risk-neutral.

Our empirical analysis is based on very detailed bidder-level data from French Treasury securities auctions held in 1995. In France, the Treasury sells the securities via discriminatory auctions. In the first part of the empirical analysis our purpose is therefore to estimate the parameters of the discriminatory auction model using the two-step estimation procedure. The second part is more policy-oriented. Given an explicit optimal strategy in the uniform auction model (derived under the same distributional assumptions as in the discriminatory model), and using the parameter estimates of the discriminatory auction model, we can approximate the hypothetical equilibrium price that would emerge in the uniform auction, and thereby also the corresponding revenue. Comparing, for each auction held in 1995, the observed revenue (from the discriminatory auction) with the hypothetical revenue generated by the uniform auction, we can evaluate if the discriminatory auction is revenue-superior to the uniform auction, or not.

We hereby contribute to a debate that has been going on at least since Friedman (1960). He supported the uniform auction format, claiming that collusion is less likely in this auction system and that for this reason the uniform auction would be better from the Treasury's viewpoint than a discriminatory auction. His article initiated a vast literature on the revenue generating properties of the uniform and discriminatory auction formats. Although most of the papers in this literature are theoretical, there are some empirical contributions. The empirical studies on the optimal revenue debate are either based on experimental data (see for example Smith, 1967, and Abbink, Brandts and Pezanis-Christou, 2001), or on natural experiment-type data (see for example Umlauf, 1993, who examines the revenue issue by exploiting the fact that the Mexican Treasury switched from discriminatory pricing to uniform pricing, and Berg, Boukai and Landsberger, 1998, who exploit the fact that the Norwegian central Bank simultaneously applied both auction rules to similar types of securities). Outside a laboratory setting, and in the absence of natural experiments—France has never experimented with the uniform auction format—it is necessary to adopt a structural form approach in order to determine the best auction mechanism for the French Treasury.

Hortaçsu (2000) is the only paper we are aware of that also addresses the revenue-issue in a structural way. He proposes a model of strategic bidding behavior in the discriminatory share auction. His model differs from Wil-

son's model because the IPV paradigm is adopted instead of the common value paradigm, and also because the prices and shares submitted by the bidders are discrete instead of continuous variables (bidders in his model are constrained to submit price/share bids on a finite grid). Using the Euler condition, he manages in an ingenious way to estimate the marginal valuations of the bidders. The estimated marginal valuations enable him to reconstruct the seller's revenue in the perfectly competitive outcome, which constitutes an upper bound to revenue from the uniform auction. Using data from the Turkish Treasury auctions, he finds that the revenue generated by the discriminatory auctions exceeds the competitive revenue, implying that in Turkey the discriminatory auction is revenue-superior to the uniform auction format.

The main advantage of Hortaçsu's approach over our's is that his method is distribution-free. In his setup the distribution function of private values does not need to be specified, whereas we rely on a parametric framework to evaluate and compare the auction performances. The major disadvantage³ of Hortaçsu's approach relatively to our's is that his methodology does not always allow to rank the auction institutions in terms of revenue. Indeed, in case the competitive outcome exceeds the revenue from the discriminatory auction, then the only conclusion that can be drawn is that both the uniform and discriminatory auctions are inferior to the competitive outcome, but nothing whatsoever can be concluded about the relative performance of the 2 auction formats. In contrast, our methodology does not suffer from this drawback since simulations under the uniform auction rule can be performed under all circumstances, and hence revenue-comparisons can always be made.

The paper proceeds as follows. Section 2 presents Wilson's (1979) share auction model, the estimation method, and the asymptotic properties of the two-step estimator. Section 3 describes the institutional background of the French Treasury securities auctions, and contains a descriptive analysis of the auction data. Section 4 presents the results, and section 5 concludes.

2 Share auction theory and the estimation method

2.1 The share auction model

This subsection presents the theory of share auctions developed by Wilson (1979). A divisible good is auctioned. There are $n \geq 2$ risk-neutral bidders. The value of the good is the same for all bidders but unknown at the start of the auction. It is assumed that the value is a realization of a random variable V , which has the distribution function $F_V(v) = \Pr(V \leq v)$.⁴ Prior to the auction, each bidder $i = 1, \dots, n$ receives a private signal about the value of the good.

³Another disadvantage is that, unlike our setup, the economic fundamental in his model (the distribution of private values), is unidentified and cannot be estimated. Admittedly, given the specific policy application in Hortaçsu's paper, knowledge of the private-value distribution is not directly relevant. However, in other types of applications, when the goal is to predict bidders' behavior under alternative auction rules, it is crucial to be able to identify and estimate economic fundamentals. Still another disadvantage is that his model is an IPV model, whereas we believe, as Bikhchandani and Huang (1993), that a common value setting is more appropriate in the context of Treasury auctions.

⁴Throughout the paper random variables are distinguished from their realizations by denoting the former by upper case letters and the latter by lower case letters.

The signal received by individual i is assumed to be a realization of the random variable S_i . As usual in common value-type auction models, the bidder's signals S_1, \dots, S_n are i.i.d. given V . The distribution function of S_i given V is thus the same for all bidders i , and is denoted $F_{S|V}(s|v) = \Pr(S_i \leq s|V = v)$. It is assumed that signal S_i is only observed by bidder i , and not by the seller or the other potential buyers. Furthermore, the number of bidders n , and the distribution functions $F_V(\cdot)$ and $F_{S|V}(\cdot|\cdot)$ are common knowledge.

Each bidder i is required to submit to the auctioneer a tender (sealed, and written) stating, for each price of the good, the desired share of the good. The price-share combinations submitted by bidder i constitute bidder's i demand function for the good. By aggregating the individual demand functions, the auctioneer can determine the equilibrium price that clears the market, i.e. the price for which aggregate demand equals one. Given the pre-announced allocation device and pricing rule, the shares are then awarded to the winners, who pay the owner of the good. In the two auction formats mentioned in the introduction, the uniform price auction and the discriminatory auction, the allocation mechanism is the same, and consists in allocating to each bidder the fraction of the good he requested at the equilibrium price. The payment rule, however, differs in the two auction formats. In the uniform price auction each winner simply pays the equilibrium price multiplied by his requested share. In the discriminatory auction each winner has to pay the area under his inverse demand curve between zero and his requested share, so that here the payment is bidder-specific.

Let $x_i(\cdot, \cdot)$ be a strategy of bidder i . A strategy is a function of the price of the good p and the signal s_i , such that when bidder i receives signal $S_i = s_i$, he submits a demand schedule specifying that at each price p he demands a share $x_i(p, s_i)$ of the good. In characterizing an optimal strategy, it is assumed that the n bidders are identical (except for the fact that they can receive different signals). Furthermore, attention is restricted to symmetric equilibria, so that $x_i(\cdot, \cdot) = x(\cdot, \cdot)$ for all i . By *optimal* strategy is meant a symmetric Bayesian Nash equilibrium of the auction game. A strategy is thus optimal if no player can deviate in a profitable way from equilibrium behavior if the other players adopt the optimal strategy.

The form of an optimal strategy depends, among other things, on the auction mechanism that is used to sell the good. Consider first the characterization of an optimal strategy in the case of a uniform auction. Let $x(\cdot, \cdot)$ now designate the optimal strategy. Suppose that all bidders except i use the strategy $x(\cdot, \cdot)$, and that i uses the strategy $y(\cdot, \cdot)$. Let p^0 denote the equilibrium price, i.e. p^0 is the price such that

$$\sum_{j \neq i} x(p^0, s_j) + y(p^0, s_i) = 1. \quad (1)$$

The above market clearing equation shows that the equilibrium price depends on the signals received by the competitors of bidder i . Since these signals are unknown, the equilibrium price is also unknown to bidder i . But since bidder i knows the distribution function from which the signals S_j , $j \neq i$, are drawn, and also the function $x(\cdot, \cdot)$, he can determine the (conditional) distribution function of the random variable P^0 . That is, he can determine

$$H(p; v, y) = \Pr\{P^0 \leq p | V = v, y(p, s_i) = y, S_i = s_i\}$$

$$\begin{aligned}
&= \Pr \left\{ \sum_{j \neq i} x(p, S_j) \leq 1 - y \mid V = v, S_i = s_i \right\} \\
&= \Pr \left\{ \sum_{j \neq i} x(p, S_j) \leq 1 - y \mid V = v \right\}.
\end{aligned}$$

When bidder i uses the strategy $y(\cdot, \cdot)$, and if the value of the good and equilibrium price are respectively v and p^0 , his profit is $(v - p^0)y(p^0, s_i)$. Bidder's i expected profit in a uniform auction is therefore

$$E \left\{ \int_0^\infty (V - p)y(p, s_i) dH(p; V, y(p, s_i)) \mid S_i = s_i \right\} \quad (2)$$

where, as the notation suggests, the expectation is with respect to V given $S_i = s_i$. The strategy $x(\cdot, \cdot)$ is indeed optimal if the maximum of (2) is attained at $y(\cdot, \cdot) = x(\cdot, \cdot)$. A solution to this maximization problem can be characterized by applying the principles of calculus of variations. Wilson has shown that the necessary condition for optimization (the Euler condition) is that for all $p \in [0, \infty)$

$$0 = E \{ (V - p) \partial H(p; V, y) / \partial p + x(p, s_i) \partial H(p; V, y) / \partial y \mid S_i = s_i \} \quad (3)$$

where the partial derivatives of H with respect to p and y are evaluated at $y = x(p, s_i)$.

Next consider the discriminatory auction. Using the same notation as above, bidder's i profit in a discriminatory auction, when he adopts the strategy $y(\cdot, \cdot)$, and when the value of the good and equilibrium price are v and p^0 , equals $(v - p^0)y(p^0, s_i) - \int_{p^0}^{p^{\max}} y(u, s_i) du$, where p^{\max} is the largest price for which demand $y(\cdot, s_i)$ is non-negative. Bidder's i expected profit is therefore

$$E \left\{ \int_0^\infty \left[(V - p)y(p, s_i) - \int_p^{p^{\max}} y(u, s_i) du \right] dH(p; V, y(p, s_i)) \mid S_i = s_i \right\}. \quad (4)$$

Wilson (1979) does not derive the Euler condition for the discriminatory share auction. We show in appendix A that the Euler condition in this case is

$$0 = E \{ (V - p) \partial H(p; V, y) / \partial p - H(p; V, y) \mid S_i = s_i \} \quad (5)$$

where the distribution H and the derivative of H are evaluated at $y = x(p, s_i)$, and the condition must hold for all $p \in [0, \infty)$. Note that this Euler condition differs from the uniform-auction Euler condition (3) only in the second term of the expectation.

Our method of statistical inference is highly facilitated by rewriting the above Euler equation. Taking the expectation with respect to V, S_i , and then integrating over p , (5) can be rewritten as (see appendix A for the proof)

$$\begin{aligned}
0 &= E \{ (n - 1) \cdot (E(V \mid S_1 = s_1, \dots, S_n = s_n) - p) \cdot \mathbf{1} \{ P^0 \leq p \} \} \\
&\quad - E \{ (p - P^0) \cdot \mathbf{1} \{ P^0 \leq p \} \}
\end{aligned} \quad (6)$$

where the first expectation is with respect to S_1, \dots, S_n (the random variable P^0 only depends on S_1, \dots, S_n), the second with respect to V given S_1, \dots, S_n , the

third with respect to P^0 , and $\mathbf{1}\{\cdot\}$ is the indicator function. The condition (6) must hold for all $p \in [0, \infty)$.

The Euler condition forms the basis of our estimation method for the discriminatory share auction model. As explained in the next subsection, our estimator is defined as the minimum of an empirical counterpart of the Euler restriction. The crucial advantage of the reformulated Euler condition (6) over the Euler condition (5) is that it is much easier to obtain an empirical counterpart of the former. This comes from the fact that (6) no longer depends on $H(\cdot; \cdot, \cdot)$, the distribution function of the market clearing price. Indeed this function (and its derivative with respect to p) is difficult to compute and evaluate because of its implicit dependence on the equilibrium strategy $x(\cdot, \cdot)$.

2.2 Estimation

2.2.1 Definition of the estimator

The estimation procedure exploits the fact that the results from several different auctions are available. Suppose there are L auctions and let l index the l -th auction. In many applications the goods sold in the different auctions are not completely identical. Also, the number of bidders typically varies from auction to auction. To capture this between-auction heterogeneity we introduce the commonly known vector of variables z_l characterizing the good sold at the l -th auction, and the number of bidders, n_l .

It is assumed that the random variables (N_l, Z_l) , $l = 1, \dots, L$, are independently and identically distributed. The value of the good in the l -th auction, V_l , is assumed to depend on Z_l but not on N_l . Similarly, the signal received by individual i in auction l , S_{il} , depends on Z_l (and on V_l) but not on N_l . Conditionally on Z_1, \dots, Z_L , the values V_1, \dots, V_L are assumed to be independently and identically distributed. Furthermore, $S_{1l}, \dots, S_{n_l l}$ are independent conditionally on (V_l, Z_l) , and the signals S_{il} and $S_{i'l'}$ are independent conditionally on Z_l and $Z_{l'}$ for all $l \neq l'$. We adopt a parametric framework. The distribution functions are specified parametrically and are thus known up to a vector of parameters. The conditional distribution of V_l given $Z_l = z$ is denoted $F_{V|Z}(\cdot|z; \theta_1)$, where θ_1 is a vector of parameters. The conditional distribution function of S_{il} given $V_l = v$ and $Z_l = z$ is denoted $F_{S|V,Z}(\cdot|v, z; \theta_2)$, where θ_2 is a vector of parameters. Given the distribution functions $F_{V|Z}(\cdot|\cdot; \cdot)$ and $F_{S|V,Z}(\cdot|\cdot, \cdot; \cdot)$, we can determine the distribution function of S_{il} given $Z_l = z$, denoted $F_{S|Z}(\cdot|z; \theta)$, where $\theta = (\theta'_1, \theta'_2)'$.

The objective is now to find an estimator of θ^0 , the true value of θ . As mentioned in the introduction, our estimation method only relies on the Euler condition of the discriminatory auction model. It is therefore not necessary to dispose of an explicit optimal strategy in the case of a discriminatory auction model. It must however be assumed that there exists an equilibrium strategy, and that all agents bid according to this strategy.

Our estimator of the parameter of interest θ^0 is a two-step estimator. Let us start by describing the first step of the estimation procedure. From now on we explicitly write the optimal strategy as a function of not only the price and the signal, but also the number of bidders, the vector of auction-characteristics, and the parameter. That is, $x(p, s, n, z; \theta^0)$ is the optimal demand for the good at price p for an individual with signal s , when the auction is attended by n bidders

and has characteristics z , and when the true value of the parameter equals θ^0 . For any given $p \in [0, \infty)$, let $G(\cdot|n, z; p)$ denote the distribution function of $x(p, S_{il}, N_l, Z_l; \theta^0)$ conditionally on $N_l = n$ and $Z_l = z$. We have

$$\begin{aligned} G(x|n, z; p) &= \Pr(x(p, S_{il}, N_l, Z_l; \theta^0) \leq x | N_l = n, Z_l = z) \\ &= \Pr(x(p, S_{il}, n, z; \theta^0) \leq x | N_l = n, Z_l = z) \\ &= \Pr(S_{il} \geq x^{-1}(x, p, n, z; \theta^0) | N_l = n, Z_l = z) \\ &= \Pr(S_{il} \geq x^{-1}(x, p, n, z; \theta^0) | Z_l = z) \\ &= 1 - F_{S|Z}(x^{-1}(x, p, n, z; \theta^0) | z; \theta^0) \end{aligned}$$

where⁵ the fourth equation follows from the assumption that S_{il} and N_l are conditionally independent. The third equation holds under the additional hypothesis that the optimal strategy $x(\cdot, s, \cdot, \cdot; \cdot)$ is a strictly decreasing function in s .⁶ It follows that

$$x^{-1}(x, p, n, z; \theta^0) = F_{S|Z}^{-1}(1 - G(x|n, z; p) | z; \theta^0). \quad (7)$$

This inverse demand function will play an important role in our estimation procedure.

For any given $p \in [0, \infty)$ the distribution function $G(\cdot|n, z; p)$ can be estimated nonparametrically from the observed share bids $x_{ilp} = x(p, s_{il}, n_l, z_l; \theta^0)$, $i = 1, \dots, n_l$, $l = 1, \dots, L$, using kernel estimation methods. A nonparametric estimate of $G(\cdot|n, z; p)$ is

$$\hat{G}(x|n, z; p) = \frac{\sum_{l=1}^L \frac{1}{n_l} \sum_{i=1}^{n_l} \mathbf{1}\{x_{ilp} \leq x\} K\left(\frac{n-n_l}{h_N}, \frac{z-z_l}{h_Z}\right)}{\sum_{l=1}^L K\left(\frac{n-n_l}{h_N}, \frac{z-z_l}{h_Z}\right)} \quad (8)$$

where $K(\cdot, \cdot)$ is a kernel and h_N and h_Z are bandwidth parameters (h_Z is actually a vector of bandwidth parameters with the same dimension as z).

To proceed, let us first rewrite once more the Euler condition (6) by incorporating the auction-specific notation and variables. For auction l with characteristics z_l and with n_l bidders, the condition becomes

$$\begin{aligned} 0 &= E\{(n_l - 1) \cdot (E(V_l | S_{1l} = s_{1l}, \dots, S_{n_l l} = s_{n_l l}, N_l = n_l, Z_l = z_l) - p) \cdot \mathbf{1}\{P_l^0 \leq p\} | N_l = n_l, Z_l = z_l\} \\ &\quad - E\{(p - P_l^0) \cdot \mathbf{1}\{P_l^0 \leq p\} | N_l = n_l, Z_l = z_l\} \end{aligned} \quad (9)$$

where the random variable P_l^0 represents the equilibrium price in auction l , and the first expectation is with respect to $S_{1l}, \dots, S_{n_l l}$ given $N_l = n_l, Z_l = z_l$. The condition must hold for all $p \in [0, \infty)$ and all $l = 1, \dots, L$.

Let us next find an empirical counterpart for the above moment condition. This is not trivial since the signals $s_{1l}, \dots, s_{n_l l}$ appearing in (9) are unobserved

⁵Note that there is the function $x(\cdot, \cdot, \cdot, \cdot; \cdot)$, the inverse function $x^{-1}(\cdot, \cdot, \cdot, \cdot; \cdot)$, and the scalar x .

⁶Given specific parametric specifications of the distributions functions $F_{V|Z}(\cdot|z; \cdot)$ and $F_{S|V,Z}(\cdot|z, \cdot; \cdot)$, section (4.2) derives an explicit optimal strategy in the uniform auction model. This strategy (shown to be the unique equilibrium in a large class of demand functions) turns out to be strictly decreasing in s . It is quite natural therefore to assume that the optimal strategy in the discriminatory auction also satisfies this property.

variables. We know however that $s_{il} = x^{-1}(x_{ilp}, p, n_l, z_l; \theta^0)$. The inverse demand function $x^{-1}(\cdot, \cdot, \cdot, \cdot; \cdot)$ is unknown, but the relationship (7) suggests that, for any given θ , it is natural to replace the inverse demand function by

$$\tilde{x}^{-1}(x, p, n, z; \theta) = F_{S|Z}^{-1}(1 - \hat{G}(x|n, z; p)|z; \theta). \quad (10)$$

This in turn suggests to replace (for any value of θ) the unobserved signal s_{il} by $\tilde{x}^{-1}(x_{ilp}, p, n_l, z_l; \theta)$, and to consider the following empirical counterpart of the r.h.s. of (9):

$$\begin{aligned} & m(x_{11p}, \dots, x_{n_L L p}, n_1, \dots, n_L, p_1^0, \dots, p_L^0, z_1, \dots, z_L, p; \theta) \\ = & \sum_{l=1}^L [(E(V_l | S_{1l} = \tilde{x}^{-1}(x_{11p}, p, n_l, z_l; \theta), \dots, S_{n_l l} = \tilde{x}^{-1}(x_{n_l l p}, p, n_l, z_l; \theta), N_l = n_l, Z_l = z_l) - p) \\ & \times (n_l - 1) 1\{p_l^0 \leq p\} - (p - p_l^0) 1\{p_l^0 \leq p\}]. \end{aligned} \quad (11)$$

Since (9) must hold for all $p \in [0, \infty)$, there is an infinity of moment conditions of the form (9), and for each of these theoretical moments there exists an empirical counterpart of the form (11). The question now arises which of the theoretical moments should be exploited in the method of statistical inference. One possibility is to use *all* moments. In this case an appropriate criterion must be defined that involves all empirical moments, and minimization of the criterion over θ then leads to the estimator. For reasons given in the next subsection, this is not the course that we shall follow.

Instead, our estimation method is based on a fixed number of moments (T). Given T values for p, p_1, \dots, p_T , the estimation method exploits that (9) must hold at these prices. It is assumed that the true value θ^0 is uniquely identified from the T moments.

The second step of our estimation procedure consists in minimizing over θ the sum of the T squared empirical moments. More precisely, the second-stage estimate of θ^0 is

$$\hat{\theta} = \text{Arg min}_{\theta} \sum_{t=1}^T m^2(x_{11p_t}, \dots, x_{n_L L p_t}, n_1, \dots, n_L, p_1^0, \dots, p_L^0, z_1, \dots, z_L, p_t; \theta). \quad (12)$$

The next subsection studies the asymptotic properties of the two-step estimator.

2.2.2 Asymptotic properties of the estimator

Newey and McFadden (1994, section 8) consider a class of estimators that are defined as the solution of a set of equations involving the observables, an unknown vector of parameters of interest, and a “first-step” estimate of a function. Since the first-step estimator is a nonparametric estimator of a function rather than an estimator of a finite-dimensional parameter, this class is referred to as the class of semiparametric two-step estimators. Newey and McFadden show that these estimators can converge at a rate equal to the root of the number of observations, even though the first-step estimator converges at a slower rate. They also give regularity conditions for asymptotic normality of the second-step estimator.

By rewriting the criterion function (12) we show in appendix C that our estimator of θ^0 belongs to the class of semiparametric two-step estimators (the

nonparametric first-step estimate being \hat{G}). By verifying in the appendix that the regularity conditions of Newey and McFadden hold, we show that our estimator is \sqrt{L} -consistent, and that it is asymptotically normally distributed. For the form of the asymptotic variance of the estimator we refer to appendix C, as it involves a lot of (too much) new notation.

We restrict ourselves to a finite number of moments T to fit into the framework of Newey and McFadden. The extension to a setting where an infinity of moment conditions is exploited is beyond the scope of this paper (see however Carrasco and Florens, 1999, for an extension of the GMM to a continuum of moments).

3 Data

3.1 The institutional setting of the French government securities auctions⁷

Since 1985 the French government securities are sold through auctions. The Treasury securities are auctioned via discriminatory auctions.⁸ The auctions are organized by the Bank of France, but all decisions regarding the dates the auctions are held, the characteristics of the securities, the quantities offered, etc., are taken by the Treasury. The three main types of French Treasury securities are:

- The *Bons du Trésor à taux Fixe et à intérêts précomptés* (the BTFs); these are tradable fixed-rate short-term discount Treasury bills with maturities of up to one year.
- The *Bons du Trésor à taux Fixe et à intérêts ANnuels* (the BTANs); these are tradable fixed-rate medium-term Treasury notes with interest paid annually and with maturities of two or five years.
- The *Obligations Assimilables du Trésor* (the OATs); these are fungible Treasury bonds with maturities ranging between 7 and 30 years.

Since our empirical analysis is based on the auctions for OATs and BTANs that were held in 1995, we describe the auction environment for these particular securities only, and present the auction rules as they prevailed at that time, even though they may have slightly changed by now.

The Treasury auctions for OATs and BTANs are held once a month—OATs on the first Thursday of the month and BTANs on the third Thursday. The scheduling of the auctions is as follows:

- Two business days prior to the auction, the Treasury announces the line that is to be auctioned, i.e. the Treasury describes the characteristics of the securities (the nominal yield, the maturity, etc.), and also determines the amount of securities it plans to sell.

⁷This subsection draws on the annual report (we use the 1995 version) on the French Government Securities. For more details see this report.

⁸Bartolini and Cottarelli (1997) find that 42 out of 77 countries in their sample (including all G-7 countries) used auctions in 1993 to sell government bills. More than 90% of the 42 countries that relied on auctions adopted the discriminatory auction format.

- A bid consists of a price/quantity pair. The price is expressed as a percentage (of the nominal value of the security), and the quantity is the amount (in FFr) of the security the bidder wants at the corresponding price.⁹ The minimal amount that bidders may submit is FFr1 million for the BTANs and FFr50 million for the OATs. Bidders are allowed to submit as many bids as they wish. Bids can be submitted until 10 minutes before the start of the auction. Most bids are submitted via TELSAT, a computerized bidding system, but bidders can also submit their proposals in sealed envelopes directly to the Bank of France. The start of the auction is at 11 am.
- The Bank of France ranks bids in descending order of price, and immediately transmits the list of ranked bids to the Treasury without revealing the identity of the bidders.
- The Treasury determines the amount of securities to be sold at auction, and calculates the stop-out price (the equilibrium price), i.e. the price for which aggregate demand equals the amount of securities offered at the auction. The highest prices are served first, and the allocation of securities stops when the amount (sold at auction) is reached. Any ties are settled on a pro rata basis.
- The auction results are announced via TELSAT (and also via some other information networks). The delay between the deadline for submission and the release of the results was less than 20 minutes in 1995.
- Delivery and settlement of the OATs take place on the 25th of the month the auction is held. In the case of 2-year (resp. 5-year) BTANs this occurs on the 5th (resp. 12th) of the month following the auction.

Most bidders are *Spécialistes en Valeurs du Trésor* (SVTs).¹⁰ The SVTs are primary dealers selected by the Treasury among the most active players on the government securities market. On average they account for 90% of the securities bought at auctions, the remaining 10% being purchased by other large banks or financial institutions. The mission of the SVTs, set out in a charter, is roughly speaking to run and maintain the primary and secondary government debt market and to advise the government on the implementation of its issuance policy.

France belongs to a small group of countries¹¹ in which bidders have the possibility to submit *Offres Non Compétitives* (ONCs). These non-competitive bids (the competitive bids being the auction bids described until now) can actually only be submitted by the SVTs. There are two kinds of non-competitive bids: the ONC1s, which must be submitted at the same time as the competitive bids, and the ONC2s, which may be submitted once the auction is over and

⁹For example, a bidder submitting (90%;FFr100 million) states that at the price of 90% he requests FFr100 million worth of securities. Re-formulated in a more familiar way, assuming the nominal value equals FFr2000, this bidder demands 50000 securities (FFr100 million divided by FFr2000) when the price per security is FFr1800 (90% of FFr2000).

¹⁰On 1 April 1995 there were 19 SVTs.

¹¹According to Bartolini and Cottarelli (1997), only 38% of 40 countries that relied on auctions accepted non-competitive bids.

until one day after the auction. The SVTs are not obliged to engage in non-competitive bidding. Each SVT can not submit more than one ONC1 bid and one ONC2 bid. Unlike a competitive bid, a non-competitive bid consists only in an amount (in FFr) of the security the bidder wants. The amount submitted by a bidder is sealed and may not exceed a certain bidder-specific limit, but except for this restriction each bidder is guaranteed the quantity he bids for. The limit may differ for the two types of ONCs, and its height is determined by the participation of the bidder in the three last auctions. The price at which the non-competitive bids must be paid is identical for all bidders and corresponds to the quantity-weighted average price of the awarded competitive bids. Since ONC1s are submitted *before* the main auction and ONC2s *after* the auction results are revealed, the former are submitted under *price-uncertainty* and the latter under *price-certainty*.

French Treasury securities can not only be bought at auction but also on two other markets: the so-called when-issued market and the secondary market (see Bikhchandani and Huang, 1993, for a detailed description of the US-version of these two market forms). The when-issued market for a security is a forward market that starts the day when the Treasury announces an auction for that security, and ends on the settlement date. Once the announcement has been made by the Treasury, dealers can trade forward contracts on the Treasury securities that are to be auctioned. Contract sellers and contract buyers commit themselves to respectively deliver and take delivery of certain specified amounts of securities at the forward prices. The forward contracts are delivered on the issue date (=settlement date) of the security, which explains why the market is called a when-issued market. The BTANs and OATs are also traded on the secondary market. The secondary market is a permanently active market where the smaller financial institutions and individual investors can trade in securities and where the competitive bidders can resale their securities obtained at auction.

3.2 The link between theory and practice

As mentioned in the introduction, Wilson’s share auction model is often regarded as a realistic model of Treasury auctions. From our description of the institutional setting of the French Treasury auctions it is clear, however, that the theory deviates in several ways from practice. The purpose of this subsection is to comment on these deviations.

The first deviation is that, unlike the share auction model, part of the bidders in France—the SVTs—have the possibility to submit ONC1s and ONC2s. The fact that bidders have this opportunity implies that their maximization problem differs from the maximization problem (4). If non-competitive bidding is allowed then agents maximize expected earnings derived from competitive *and* non-competitive bidding, and consequently the actual optimal bidding strategies may differ from those derived in subsection (2.1). The reason why we have not attempted to extend Wilson’s model to account for non-competitive bidding is that this phenomenon is not important in France. Indeed, the descriptive statistics in the next subsection (Table 1) indicate that on average ONC1s (ONC2s) only amount to 1% (8%) of the total amount of securities issued by the Treasury (that is the amount sold at auction plus the ONC1s plus the ONC2s). Apparently bidders in France do not make much use of non-competitive bidding, and

the effect of neglecting non-competitive bidding in our analysis is therefore expected to be negligible.

The second deviation is that unlike most Treasury auctions in the world (including the BTF auctions in France), the French Treasury does not announce the precise amount of OATs or BTANs it plans to sell. Instead it announces an issue interval, wherein the quantity of securities eventually sold at auction necessarily lies. Ex-ante this induces some uncertainty about the total amount of securities sold at auction. However, in practice the intervals announced by the Treasury are quite tight, so that we have not attempted to allow for supply uncertainty in the model.¹²

The third deviation is that Wilson’s model describes the main auction as an isolated market, unaffected by what happens on the when-issued market and the secondary market. The interdependencies that exist between the three market forms might affect bidder’s behavior at the auction. However, we feel that it is beyond the scope of this paper to construct (and estimate) a unified model that integrates the three markets for Treasury securities.

3.3 Descriptive analysis

Our empirical analysis is based on all French-franc denominated OAT and BTAN auctions that were held in 1995.¹³ Table A1 in the appendix gives the auction dates, the lines auctioned (this column in the table gives for each auction the nominal yield of the security and the year of maturity), the settlement dates, and the exact maturity dates. As mentioned in subsection (3.1), OATs and BTANs were auctioned once per month—OATs on the first Thursday of the month and BTANs on the third Thursday of the month. There are therefore 24 different auction dates. As table A1 indicates, the Treasury did not necessarily sell just one line on a given auction date, but often sold two or even three lines on the same day. If several lines were offered on a given day, they were sold simultaneously but via strictly separate auctions.¹⁴ Six different lines of OATs and five different lines of BTANs were issued in 1995.

Apart from the information already given in table A1, we observe for each auction all competitive bids submitted by all bidders (the non-competitive bids are not observed in our data), the stop-out price, the amount of securities sold at auction, and the total amounts of ONC1s and ONC2s awarded by the Treasury. The auction participants are unidentified in our data, i.e. we cannot tell from the data which bidder in a given auction corresponds to which bidder in some other auction.

Table 1 gives some overall information about the auctions. In 1995 a total of 45 auctions took place: 25 OAT auctions and 20 BTAN auctions. The total

¹²Note that Wang and Zender (2002) have generalized Wilson’s model by allowing for supply uncertainty.

¹³Three other types of French government securities were issued through auctions by the Treasury in 1995: BTFs, ECU-denominated OATs/BTANs, and OATs for private investors. We have excluded the data from these auctions for homogeneity reasons. Indeed, the auction rules under which they were conducted differed from the rules under which the auctions in our sample were held.

¹⁴For example, the investors interested in both lines issued on May 1995—“7.75% April 2005” OATs and “8.5% October 2008” OATs—had to submit their bids separately for the two auctions, according to the procedures described in subsection (3.1), and at 11 am the two lines were auctioned independently from each other.

quantity issued by the Treasury in the 45 auctions is FFr464.579 billion, so the mean amount of securities sold is FFr10.324 billion per auction. The total quantity can be split up into the total amount of awarded competitive bids (FFr 423.72 billion), total amount of awarded ONC1s (FFr4.831 billion), and total of awarded ONC2s (FFr36.028 billion). A total of 937 “different ” bidders have participated in the 45 auctions, and the total number of competitive bids submitted by these bidders is 2677. About 38% of these 2677 bids were served by the Treasury, 16% were only partially served (because they were at the stop-out price), and almost half of the bids (46%) were losing bids.

Table 1. Overall information about the auctions

| | |
|----------------------------------------------------|---------------|
| Number of auctions | 45 |
| OAT | 25 (56%) |
| BTAN | 20 (44%) |
| Number of bidders | 937 |
| Number of bids | 2677 |
| Totally served | 1 016 (38%) |
| Partially served | 423 (16%) |
| Not served | 1 238 (46%) |
| Total amount issued by the Treasury (FFr millions) | 464 579 |
| competitive bids (FFr millions) | 423 720 (91%) |
| ONC1 (FFr millions) | 4 831 (1%) |
| ONC2 (FFr millions) | 36 028 (8%) |

Table 2 presents summary statistics per auction. The average number of bidders per auction is 20.82. The number of bidders is quite stable across auctions and is close to the total number of SVTs in 1995. The number of bids per auction ranges between 28 and 102 bids, and the mean is about 60 bids. The mean of the auction coverage—which is the ratio of the sum of all submitted competitive bids and ONC1s to the total amount served (the awarded competitive bids and ONC1s)—is equal to 2.25.

Table 2. Summary statistics per auction

| Variables | Mean | Std. dev. | Min | Max | Obs |
|--------------------------------------------------|--------|-----------|-------|--------|-----|
| Number of bidders | 20.82 | 1.71 | 15 | 23 | 45 |
| Number of bids | 59.49 | 17.41 | 28 | 102 | 45 |
| Amount issued by Treasury (FFr millions) | 10 324 | 5 922 | 2 052 | 21 849 | 45 |
| Winning competitive bids (FFr millions) | 9 416 | 5 335 | 1 800 | 19 125 | 45 |
| ONC1 (FFr millions) | 107 | 121 | 0 | 496 | 45 |
| ONC2 (FFr millions) | 801 | 820 | 0 | 2 553 | 45 |
| Auction coverage | 2.25 | 0.75 | 1.29 | 5.18 | 45 |
| Maturity of security (in days) | 3 749 | 3 227 | 586 | 11 231 | 45 |
| Nominal yield (%) | 7.31 | 0.80 | 5.75 | 8.50 | 45 |
| Secondary market price | 98.07 | 9.29 | 71.33 | 108.50 | 45 |
| Stop-out price | 97.94 | 9.40 | 70.88 | 108.16 | 45 |
| Highest price bid - lowest price bid | 0.32 | 0.13 | 0.10 | 0.68 | 45 |
| Auction scatter (average price - stop-out price) | 0.03 | 0.02 | 0.00 | 0.16 | 45 |

The mean of the maturity of the security—defined as the number of days between the settlement date and the date of maturity—is 3 749 days. The average nominal yield in the sample is 7.31%. The average of the secondary market price—defined as the opening secondary market price of the security on the day of the auction—equals 98.07. The stop-out price varies between 70.88 and 108.16, and has a mean equal to 97.94. The dispersion of the submitted prices—defined as the highest price minus the lowest price—is on average 0.32. Finally, the mean of the auction scatter—still another measure of the dispersion of auction prices, and defined as the difference between the weighted average price of the winning bids and the stop-out price—equals 0.03.

Table 3 gives summary statistics per bidder or per bid. The number of submissions per bidder ranges from 1 to 9, and the mean equals 2.86. This is slightly lower than the mean of 3.2 bids found by Gordy (1999) (based on data from Portuguese Treasury auctions), but substantially lower than the mean of 6.9 bids found by Hortaçsu (2000) (Turkish Treasury auctions).

Table 3. Summary statistics per bidder or per bid

| Variable | Mean | Std. dev. | Min | Max | Obs |
|------------------------------------------|-------|-----------|-------|--------|------|
| Number of bids | 2.86 | 1.58 | 1 | 9 | 937 |
| Demanded quantity per bid (FFr millions) | 326 | 328 | 10 | 2500 | 2677 |
| Price bid | 98.54 | 7.93 | 70.54 | 108.26 | 2677 |
| Highest price bid - lowest price bid | 0.07 | 0.07 | 0 | 0.54 | 937 |

In our sample the submitted quantities range from FFr10 million to FFr2.5 billion, and prices from 70.54 to 108.26. The price dispersion per bidder ranges between 0 (if the bidder has submitted only one bid) and 0.54.

4 Results

4.1 Estimation of the parameters of the discriminatory model

The secondary market price, the nominal yield and the maturity of the security (divided by 1000) sold at the l -th auction are the variables included in the vector z_l . The dimension of z_l is thus equal to 3, and we denote $z_l = (z_{1l}, z_{2l}, z_{3l})$. In the first step of our estimation procedure we nonparametrically estimate the distribution function $G(x|n, z; p)$ using the Epanechnikov kernel. To avoid any confusion with the vector z_l , we denote the 3-dimensional vector of explanatory variables at which $G(\cdot|\cdot, \cdot; \cdot)$ is evaluated, as $z = (z^1, z^2, z^3)$. In expression (8) we thus have $K\left(\frac{n-n_l}{h_N}, \frac{z-z_l}{h_Z}\right) = K\left(\frac{n-n_l}{h_N}\right) K\left(\frac{z^1-z_{1l}}{h_{1Z}}\right) K\left(\frac{z^2-z_{2l}}{h_{2Z}}\right) K\left(\frac{z^3-z_{3l}}{h_{3Z}}\right)$ where $K(u) = 0.75(1-u^2)1\{|u| \leq 1\}$, and h_N, h_{1Z}, h_{2Z} and h_{3Z} are the bandwidth parameters. The choice of all bandwidth parameters were chosen according to the rule of thumb defining each bandwidth as 2.214 multiplied by the standard error of the variable multiplied the number of observations (L) to the power $-\frac{1}{7}$.¹⁵ We find $h_N = 2.2$, $h_{1Z} = 11.9$ (bandwidth of the secondary mar-

¹⁵Newey and McFadden (1994, pp. 2203-2210) impose conditions on the choice of the kernel and the convergence rate of the bandwidth parameters. To satisfy these conditions, we chose

ket price), $h_{2Z} = 1.0$ (nominal yield), and $h_{3Z} = 4.1$ (maturity of the security divided by 1000).

To proceed with the second step, we have to choose parametric specifications for the distribution functions of the signal and the value. For the revenue comparison in the next subsection, the specifications should be chosen such that explicit optimal strategies can be obtained in the uniform share auction model. Bearing this in mind, we assume that the value V_l given $Z_l = z_l$ has the distribution function

$$F_{V_l|Z}(v|z_l; \theta_1) = \int_0^v \gamma u^{\gamma-1} \frac{\beta_l^{\alpha_l}}{\Gamma(\alpha_l)} u^{\gamma(\alpha_l-1)} \exp[-\beta_l u^\gamma] du \quad (13)$$

where

$$\begin{aligned} \alpha_l &= (1, z_l) \cdot \alpha \\ \beta_l &= (1, z_l) \cdot \beta \end{aligned}$$

and $\Gamma(\cdot)$ is the gamma function, α and β are vectors (of dimension 4 by 1) of parameters, and γ is a scalar parameter. Note that if $\gamma = 1$ then the above distribution function corresponds to the gamma distribution function with parameters α_l and β_l , i.e. in this case V_l follows a gamma distribution with conditional mean α_l/β_l and conditional variance α_l/β_l^2 ; if $\gamma \neq 1$ then V_l^γ is distributed as a gamma distribution with parameters α_l and β_l ; if $\alpha_l = 1$ then V_l follows a weibull distribution with parameters γ and β_l . Note also that $\theta'_1 = (\alpha', \beta', \gamma)$.

We furthermore assume that the signal S_{il} given $V_l = v_l$ and $Z_l = z_l$ follows an exponential distribution:

$$F_{S_{il}|V,Z}(s|v_l, z_l; \theta_2) = 1 - \exp[-sv_l^\gamma] \quad (14)$$

where γ is the scalar parameter that also appears in the conditional distribution function of V_l . Note that the conditional expectation and variance of S_{il} are assumed to be independent of z_l . Note also that $\theta_2 = \gamma$, so the complete vector of parameters is therefore $\theta' = (\alpha', \beta', \gamma)$.

In the second step of the estimation procedure we estimate θ^0 , the true value of θ . The estimate of this parameter is defined by (12), where, given the specifications (13) and (14), the conditional expectation of V_l appearing in the empirical moment $m(\cdot)$ is:

$$\begin{aligned} E(V_l|S_{1l} = \tilde{x}^{-1}(x_{1lp}, p, n_l, z_l; \theta), \dots, S_{n_l l} = \tilde{x}^{-1}(x_{n_l lp}, p, n_l, z_l; \theta), N_l = n_l, Z_l = z_l) \\ = \frac{\Gamma(n_l + \alpha_l + 1/\gamma)}{\Gamma(n_l + \alpha_l)} \frac{1}{\left(\beta_l + \sum_{i=1}^{n_l} \beta_l \left[\frac{1}{\widehat{G}^{1/\alpha_l}(x_{ilp}|n_l, z_l; p)} - 1 \right] \right)^{1/\gamma}}. \end{aligned} \quad (15)$$

We choose $T = 45$, i.e. the number of moments equals the number of auctions in the sample, and the prices p_1, \dots, p_T are equal to the observed stop-out prices.

The second-step estimates are presented in Table 4, together with the estimated standard errors (using the estimated asymptotic variance matrix of $\hat{\theta}$ given in the appendix).

the Epanechnikov kernel ($m = 2$ in the notation of Newey and McFadden's lemma 8.10, p. 2206), and the convergence rate $L^{-\frac{1}{4}}$. Furthermore in this formula, the factor 2.214 is the constant associated with the Epanechnikov kernel.

Table 4. Second-step estimate of θ (est. standard error)

| | |
|-------------------------------------|----------------------|
| Estimate of α : | |
| Constant | -15848.93 (320.72)** |
| Secondary market price | 66.67 (3.48)** |
| Nominal yield | 1617.29 (8.76)** |
| Maturity of security (in days/1000) | 142.24 (6.96)** |
| Estimate of β : | |
| Constant | 8596.67 (114.87)** |
| Secondary market price | -104.30 (1.24)** |
| Nominal yield | 340.41 (6.77)** |
| Maturity of security (in days/1000) | -2.04 (1.57) |
| γ | 12.28 (0.0085)** |

Note: Actual estimates and standard errors in β are very small, and those reported in the table are multiplied by 10^{24} .

All parameters are significantly different from zero at the 5% level, except the parameter corresponding to the maturity of the security in β . Given the estimate $\hat{\theta}$ and using (13), we can calculate $E[V_l|Z_l = z_l]$, and the derivative of this expectation with respect to each variable in z_l . Evaluated at the empirical mean of the explanatory variables (i.e. we replace z_l by the averages reported in Table 2), we find that the conditional mean of the value equals 99.79, which is above the average secondary market price (98.07) and the average stop-out-price (97.94). The derivative with respect to the nominal yield, the secondary market price and the maturity are 1.06, 1.18 and 0.40 respectively. All these variables thus have a positive effect on the expected value of the security, as intuition suggests. Table A2 in the appendix gives for all auctions $l = 1, \dots, 45$, the estimated expectations $E[V_l|S_{1l} = \hat{s}_{1l}, \dots, S_{nl} = \hat{s}_{nl}, Z_l = z_l]$ (with the estimated signal $\hat{s}_{il} = F_{S|Z}^{-1}(1 - \hat{G}(x_{ilp_l^0}|n_l, z_l; p_l^0)|z_l; \hat{\theta})$, and z_l the value of the characteristics of the security at the l -th auction) and $E[V_l|Z_l = z_l]$, the stop-out-price, and the secondary market price.

4.2 Revenue comparison

In this subsection we compare the actual income of the French Treasury with the hypothetical income the Treasury would have earned had it adopted the uniform share auction mechanism. The actual revenue from a given discriminatory auction simply equals the sum, over all winning bids in the auction, of awarded quantities multiplied by the associated prices. The total actual income earned by the Treasury is then the sum, over the 45 auctions held in 1995, of these actual revenues. In 1995 the total actual income thus calculated amounts to FF421.543 billion.

The calculation of the hypothetical total income under the uniform auction format is less straightforward. First we need to determine an explicit optimal bidding strategy in the uniform auction format. Given our parametric specifications of the distribution functions (13) and (14), we show in appendix B that

an optimal strategy in the uniform auction is (i.e. a solution of (3))¹⁶

$$x(p, s_{il}, n_l, z_l; \theta) = \left[1 - \left\{ \frac{\beta_l}{n_l} + s_{il} \right\} \left\{ \frac{\Gamma(n_l + \alpha_l)}{\Gamma(n_l + \alpha_l + 1/\gamma)} \frac{1 + \gamma}{\gamma} p \right\}^\gamma \right] / (n_l - 1). \quad (16)$$

As mentioned in subsection 2.2.1, this optimal bidding function is decreasing in the signal s_{il} (it is also decreasing in p). In appendix B we also show that given our distributional assumptions, (16) is actually the unique equilibrium strategy in the class of demand functions of the form $x(p, s_{il}, n_l, z_l; \theta) = a(p, n_l, z_l; \theta) + b(p, n_l, z_l; \theta)s_{il}$. There are no restrictions on the functions $a(\cdot, \cdot, \cdot; \cdot)$ and $b(\cdot, \cdot, \cdot; \cdot)$ except that they must be such that $x(\cdot, \cdot, \cdot; \cdot)$ is decreasing in p and s_{il} .

The strategy (16) is a generalization of the equilibrium strategy derived in Wilson (1979, example 1). Wilson assumes that V_l follows a gamma distribution with parameters α_l and β_l , and that S_{il} is exponentially distributed with parameter v_l . His setup thus corresponds to the special case where the parameter γ appearing in the distribution functions (13) and (14) is equal to 1. When $\gamma = 1$, (16) reduces to

$$x(p, s_{il}, n_l, z_l; \theta) = \left[1 - 2p \frac{\beta_l + n_l s_{il}}{n_l(n_l + \alpha_l)} \right] / (n_l - 1) \quad (17)$$

which is the optimal bidding function given by Wilson.¹⁷

Now that an optimal bidding strategy has been determined, a possible way to calculate the hypothetical income under the uniform auction format is as follows. Given the two-step estimate $\hat{\theta}$ and using (10), define, for each bidder i and auction l , the estimated signal $\hat{s}_{il} = F_{S|Z}^{-1}(1 - \hat{G}(x_{ilp}|n_l, z_l; p)|z_l; \hat{\theta})$. Replacing, in (16), θ by $\hat{\theta}$, and for all i s_{il} by \hat{s}_{il} , gives the estimated uniform demand functions for all bidders i in auctions l . The estimated stop-out price in the l -th uniform auction can then be determined by equating estimated aggregate demand and total supply (i.e. the amount of securities sold at the l -th discriminatory auction). The hypothetical revenue from auction l is then the product of total supply and the estimated stop-out price, and the total hypothetical income under the uniform auction follows from summation over all 45 auctions in the sample.

We will however calculate the hypothetical total income in a different and more direct way by exploiting the fact that for our uniform share auction model there is an explicit solution for the stop-out price. Under the assumption that agents bid according to the optimal demand function (16), it can be shown that the stop-out price in the l -th uniform auction satisfies

$$\begin{aligned} p_l^0 &= \frac{1}{1 + 1/\gamma} E(V_l | S_{1l} = s_{1l}, \dots, S_{n_l l} = s_{n_l l}, Z_l = z_l) \\ &= \frac{1}{1 + 1/\gamma} \frac{\Gamma(n_l + \alpha_l + 1/\gamma)}{\Gamma(n_l + \alpha_l)} \frac{1}{(\beta_l + \sum_{i=1}^{n_l} s_{il})^{1/\gamma}}. \end{aligned} \quad (18)$$

¹⁶Of course, as with the Euler condition of the discriminatory model, the auction-specific notation and variables should be incorporated in the uniform Euler condition (3).

¹⁷In Wilson (1979) the numerator is $\beta_l - n_l s_{il}$ instead of $\beta_l + n_l s_{il}$. But this simply reflects a difference in the choice of the conditional distribution function of the signal. In Wilson the conditional distribution function is $F_{S|V,Z}(s_{il}|v_l, z_l; \theta_2) = e^{v_l s_{il}}$ for $s_{il} \leq 0$.

The equation for the stop-out price given by Wilson (1979, page 682) corresponds to (18) with $\gamma = 1$.¹⁸

In order to calculate the hypothetical uniform revenue we construct, as above, the estimated signals \hat{s}_{il} . The estimated stop-out price in auction l can be obtained directly on replacing θ and s_{il} by their estimates in (18). The hypothetical income in auction l and the total hypothetical income are then calculated as above.¹⁹

The hypothetical uniform revenue we calculate in this manner equals FFr400.421 billion. Therefore, had the French Treasury adopted the uniform auction format instead of the discriminatory auction format, it would have earned FFr21.122 billion less (5% of total income in the discriminatory auctions). In calculating the variance of the above estimate, we use the fact that the hypothetical revenue is simply the sum over all auctions l of p_l^0 times the amount of securities sold in the l -th auction. Hence, the estimated hypothetical revenue is some function of $\hat{\theta}$ and (via the estimated signals \hat{s}_{il}) $\hat{G}(\cdot)$, so its variance can be calculated by applying the delta method. In applying the delta method we ignore the variance in $\hat{G}(\cdot)$, i.e. we consider it as fixed. The estimated standard error we find is FFr1.11 billion, and the 95% confidence interval for the hypothetical revenue is [FFr398.210 billion; FFr402.632 billion], implying that the difference in revenue between the discriminatory and uniform auction is significant at the 5% level.

When we restrict $\gamma = 1$, the results are very different: the estimated revenue under the uniform auction drops to FFr 215.462 billion, implying a substantial loss in income of FFr206.081 billion (almost 49%). However, since the hypothesis $\gamma = 1$ is rejected by the data (see Table 4), these last figures have no statistical justification, and cannot therefore be treated without much suspicion. They merely show that Wilson's model is too restrictive for the analysis of our data, and leads to overly negative conclusions regarding the performance of the uniform auction format.

As in Hortaçsu (2000), we find that the discriminatory auction is revenue-superior to the uniform auction. Our estimated revenue loss of 5% is smaller though than the revenue loss obtained by Hortaçsu using Turkish Treasury auctions. He reports counterfactual revenue comparisons for each of the 25 auctions in his sample. His *ex-ante* revenue differences vary between 0.12% and 27%, and on average the uniform auction generated 14% less than the discriminatory auction.

5 Conclusion

This paper has proposed structural econometric methods for the empirical study of Wilson's share auction model. We have shown how the parameters of this model, i.e. the joint distribution function of the value of the good and the signals received by the bidders, can be estimated via a two-step estimation procedure. Using the estimation theory for semiparametric two-step estimators developed by Newey and McFadden (1994), we have established the asymptotic properties

¹⁸Except that the denominator in the last term equals $\beta_l - \sum_{i=1}^{n_l} s_{il}$ instead of $\beta_l + \sum_{i=1}^{n_l} s_{il}$.

¹⁹The total hypothetical income has been calculated in two ways. First by estimating the signals as $\hat{s}_{il} = F_{S|Z}^{-1}(1 - \hat{G}(x_{il} p_l^0 | n_l, z_l; p_l^0) | z_l; \hat{\theta})$, and second by estimating the signals as an average $\hat{s}_{il} = \frac{1}{L} \sum_{l'=1}^L F_{S|Z}^{-1}(1 - \hat{G}(x_{il} p_{l'}^0 | n_l, z_l; p_{l'}^0) | z_l; \hat{\theta})$. The estimated hypothetical income is not affected by the manner in which the signals are estimated.

of our estimator. A crucial feature of the method of statistical inference is that it only relies on the first-order condition of the bidder's maximization problem. Our estimation method seems therefore potentially of interest whenever one wants to estimate game theoretic models for which no explicit strategies can be found.

The methods have been applied to Treasury auctions held in France. Our results suggest that the Treasury's revenue in the discriminatory share auction is 5% higher than in the uniform share auction, which is a relatively high figure given the enormous amounts of money at stake. This result can be seen as an ex-post justification for the fact that the majority of countries rely on the discriminatory auctions to sell their Treasury securities.

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APPENDIX

A The Euler condition for the discriminatory auction model

First we rewrite the expected profit (4). We have

$$\begin{aligned}
& E \left\{ \int_0^\infty \left[(V-p)y(p, s_i) - \int_p^{p^{\max}} y(u, s_i) du \right] dH(p; V, y(p, s_i)) | S_i = s_i \right\} \\
&= E \left\{ - \int_0^\infty [-y(p, s_i) + (V-p)y_p(p, s_i) + y(p, s_i)] H(p; V, y(p, s_i)) dp | S_i = s_i \right\} \\
&= E \left\{ \int_0^\infty [-(V-p)y_p(p, s_i) H(p; V, y(p, s_i))] dp | S_i = s_i \right\}
\end{aligned}$$

where the first equality follows from an integration by parts, and y_p is the derivative of y with respect to p . The expression in brackets can be written as a function $g(p, y(p, s_i), y_p(p, s_i), V)$. We want to stress that there is no connection at all between this function g and the distribution function G defined in the main text. Using this notation, the expected profit can be rewritten as

$$E \left\{ \int_0^\infty [g(p, y(p, s_i), y_p(p, s_i), V)] dp | S_i = s_i \right\}. \quad (19)$$

The necessary condition for $y(\cdot, s_i)$ to maximize (19) is that for all $p \in [0, \infty)$

$$0 = E \left\{ \frac{\partial g}{\partial y} - \frac{d}{dp} \frac{\partial g}{\partial y_p} | S_i = s_i \right\} \quad (20)$$

(see Chiang, 1992, p. 46). Given the specific form of the function g , we have

$$\frac{\partial g}{\partial y} = -(V-p)y_p(p, s_i) \frac{\partial H(p; V, y(p, s_i))}{\partial y}$$

and

$$\frac{d}{dp} \frac{\partial g}{\partial y_p} = H(p; V, y(p, s_i)) - (V-p) \left\{ \frac{\partial H(p; V, y(p, s_i))}{\partial p} + y_p(p, s_i) \frac{\partial H(p; V, y(p, s_i))}{\partial y} \right\}$$

so that (20) can be rewritten as

$$0 = E \left\{ -H(p; V, y(p, s_i)) + (V-p) \frac{\partial H(p; V, y(p, s_i))}{\partial p} | S_i = s_i \right\}.$$

The strategy $x(\cdot, \cdot)$ is optimal if the above condition is satisfied for $y(\cdot, s_i) = x(\cdot, s_i)$, which gives the Euler condition (5).

Next we show that the Euler condition (5) can be rewritten as (6). We have

$$H(p; v, y) = \int \dots \int_{s_j; j \neq i} \mathbf{1} \left\{ \sum_{j \neq i} x(p, s_j) \leq 1 - y \right\} \prod_{j \neq i} f_{S|V}(s_j | v) ds_j$$

where $f_{S|V}(\cdot|\cdot)$ is the density associated with $F_{S|V}(\cdot|\cdot)$. Defining $x_{jp} = x(p, s_j)$, the above expression can be written as

$$H(p; v, y) = \int \dots \int_{x_{jp}; j \neq i} \mathbf{1} \left\{ \sum_{j \neq i} x_{jp} \leq 1 - y \right\} \prod_{j \neq i} \frac{\partial x^{-1}(p, x_{jp})}{\partial x_{jp}} f_{S|V}(x^{-1}(p, x_{jp})|v) dx_{jp}$$

so that (because the integrand is symmetric in all the x_{jp})

$$\begin{aligned} \frac{\partial H(p; v, y)}{\partial p} &= (n-1) \int \dots \int_{x_{jp}; j \neq i} \mathbf{1} \left\{ \sum_{j \neq i} x_{jp} \leq 1 - y \right\} \\ &\quad \times \frac{\partial}{\partial p} \left[\frac{\partial x^{-1}(p, x_{1p})}{\partial x_{1p}} f_{S|V}(x^{-1}(p, x_{1p})|v) \right] dx_{1p} \\ &\quad \times \prod_{\substack{j \neq i \\ j \neq 1}} \frac{\partial x^{-1}(p, x_{jp})}{\partial x_{jp}} f_{S|V}(x^{-1}(p, x_{jp})|v) dx_{jp}. \end{aligned}$$

Now define $H(p; v) = \Pr(P^0 \leq p | V = v)$ and $H(p) = \Pr(P^0 \leq p)$. We have

$$\begin{aligned} H(p; v) &= \int \dots \int \mathbf{1} \left\{ \sum_{j=1}^n x(p, s_j) \leq 1 \right\} \prod_{j=1}^n f_{S|V}(s_j|v) ds_j \\ &= \int \dots \int \mathbf{1} \left\{ \sum_{j=1}^n x_{jp} \leq 1 \right\} \prod_{j=1}^n \frac{\partial x^{-1}(p, x_{jp})}{\partial x_{jp}} f_{S|V}(x^{-1}(p, x_{jp})|v) dx_{jp}. \end{aligned}$$

So, again by symmetry, we have

$$\begin{aligned} \frac{dH(p; v)}{dp} &= n \int \dots \int \mathbf{1} \left\{ \sum_{j=1}^n x_{jp} \leq 1 \right\} \frac{\partial}{\partial p} \left[\frac{\partial x^{-1}(p, x_{1p})}{\partial x_{1p}} f_{S|V}(x^{-1}(p, x_{1p})|v) \right] dx_{1p} \\ &\quad \times \prod_{j=2}^n \frac{\partial x^{-1}(p, x_{jp})}{\partial x_{jp}} f_{S|V}(x^{-1}(p, x_{jp})|v) dx_{jp}. \end{aligned}$$

After some straightforward calculations it follows that

$$\begin{aligned} E \left\{ \frac{\partial H(p; v, x(p, S_i))}{\partial p} | V = v \right\} &= \int \frac{\partial H(p; v, x(p, s_i))}{\partial p} f_{S|V}(s_i|v) ds_i \\ &= \frac{(n-1)}{n} \frac{dH(p; v)}{dp} \end{aligned}$$

and therefore

$$E \left\{ \frac{\partial H(p; V, x(p, S_i))}{\partial p} \right\} = \frac{(n-1)}{n} \frac{dH(p)}{dp}.$$

We also have

$$\begin{aligned} E \left\{ V \frac{\partial H(p; V, x(p, S_i))}{\partial p} \right\} &= \frac{(n-1)}{n} \int v f_V(v) \frac{dH(p; v)}{dp} dv \\ &= \frac{(n-1)}{n} \frac{d}{dp} \left[\int v f_V(v) \int \dots \int \mathbf{1} \{P^0 \leq p\} \prod_{j=1}^n f_{S|V}(s_j|v) ds_j dv \right] \\ &= \frac{(n-1)}{n} \frac{d}{dp} [E (E(V | S_1 = s_1, \dots, S_n = s_n) \cdot \mathbf{1} \{P^0 \leq p\})] \end{aligned}$$

where $f_V(\cdot)$ is the density associated with $F_V(\cdot)$. Finally we have $E\{H(p; V, x(p, S_i))\} = H(p)$.

Therefore, taking the expectation with respect to V, S_i , the Euler condition (5) can be rewritten as

$$\begin{aligned} 0 &= E\left\{V\frac{\partial H(p; V, x(p, S_i))}{\partial p}\right\} - pE\left\{\frac{\partial H(p; V, x(p, S_i))}{\partial p}\right\} - E\{H(p; V, x(p, S_i))\} \\ &= \frac{(n-1)}{n}\frac{d}{dp}\left[E(E(V|S_1 = s_1, \dots, S_n = s_n) \cdot \mathbf{1}\{P^0 \leq p\})\right] \\ &\quad - p\frac{(n-1)}{n}\frac{dH(p)}{dp} - H(p). \end{aligned}$$

Integrating over p gives

$$\begin{aligned} C &= E\{(n-1) \cdot (E(V|S_1 = s_1, \dots, S_n = s_n) - p) \cdot \mathbf{1}\{P^0 \leq p\}\} \\ &\quad - E\{(p - P^0) \cdot \mathbf{1}\{P^0 \leq p\}\} \end{aligned}$$

where C is the integration constant. We now assume that $\Pr(P^0 \leq 0) = 0$. Replacing $p = 0$ in the above expression, it follows then that $C = 0$, which gives the condition (6).

B An optimal strategy for the uniform auction model

In this appendix we prove the claim that (16) is the unique equilibrium strategy belonging to the class of demand functions of the form $x(p, s_{il}, n_i, z_i; \theta) = a(p, n_i, z_i; \theta) + b(p, n_i, z_i; \theta)s_{il}$.

First we derive the conditional expectation $E(V_i|\tilde{S}_i = \tilde{s}_i, S_{il} = s_{il})$ where $\tilde{S}_i \equiv \sum_{j \neq i} S_{jl}$. In this section of the appendix a density function (marginal, joint, or conditional) is represented by the letter g . Again, there is no connection between this function g and the distribution function G defined in the main text. We have

$$\begin{aligned} E(V_i|\tilde{S}_i &= \tilde{s}_i, S_{il} = s_{il}) \\ &= \int_0^\infty v_l g(v_l|\tilde{s}_i, s_{il}) dv_l \\ &= \frac{\int_0^\infty v_l g(\tilde{s}_i|v_l) g(s_{il}|v_l) g(v_l) dv_l}{\int_0^\infty g(\tilde{s}_i|v_l) g(s_{il}|v_l) g(v_l) dv_l} \\ &= \frac{\int_0^\infty v_l \frac{v_l^{\gamma(n_i-1)}}{\Gamma(n_i-1)} \tilde{s}_i^{n_i-2} e^{-v_l^\gamma \tilde{s}_i} v_l^\gamma e^{-v_l^\gamma s_{il}} \frac{\beta_i^{\alpha_i}}{\Gamma(\alpha_i)} v_l^{\gamma(\alpha_i-1)} e^{-\beta_i v_l^\gamma} \gamma v_l^{\gamma-1} dv_l}{\int_0^\infty \frac{v_l^{\gamma(n_i-1)}}{\Gamma(n_i-1)} \tilde{s}_i^{n_i-2} e^{-v_l^\gamma \tilde{s}_i} v_l^\gamma e^{-v_l^\gamma s_{il}} \frac{\beta_i^{\alpha_i}}{\Gamma(\alpha_i)} v_l^{\gamma(\alpha_i-1)} e^{-\beta_i v_l^\gamma} \gamma v_l^{\gamma-1} dv_l} \\ &= \frac{\int_0^\infty v_l^{1+\gamma(n_i+\alpha_i-1)} e^{-(\tilde{s}_i+\beta_i+s_{il})v_l^\gamma} \gamma v_l^{\gamma-1} dv_l}{\int_0^\infty v_l^{\gamma(n_i+\alpha_i-1)} e^{-(\tilde{s}_i+\beta_i+s_{il})v_l^\gamma} \gamma v_l^{\gamma-1} dv_l} \\ &= \frac{\int_0^\infty v_l^{n_i+\alpha_i-1+1/\gamma} e^{-(\tilde{s}_i+\beta_i+s_{il})v_l} dv_l}{\int_0^\infty v_l^{n_i+\alpha_i-1} e^{-(\tilde{s}_i+\beta_i+s_{il})v_l} dv_l} \end{aligned}$$

$$\begin{aligned}
&= \frac{\frac{\Gamma(n_l + \alpha_l + 1/\gamma)}{(\tilde{s}_l + \beta_l + s_{il})^{n_l + \alpha_l + 1/\gamma}} \int_0^\infty \frac{(\tilde{s}_l + \beta_l + s_{il})^{n_l + \alpha_l + 1/\gamma}}{\Gamma(n_l + \alpha_l + 1/\gamma)} v_l^{n_l + \alpha_l - 1 + 1/\gamma} e^{-(\tilde{s}_l + \beta_l + s_{il})v_l} dv_l}{\frac{\Gamma(n_l + \alpha_l)}{(\tilde{s}_l + \beta_l + s_{il})^{n_l + \alpha_l}} \int_0^\infty \frac{(\tilde{s}_l + \beta_l + s_{il})^{n_l + \alpha_l}}{\Gamma(n_l + \alpha_l)} v_l^{n_l + \alpha_l - 1} e^{-(\tilde{s}_l + \beta_l + s_{il})v_l} dv_l} \\
&= \frac{\Gamma(n_l + \alpha_l + 1/\gamma)}{(\tilde{s}_l + \beta_l + s_{il})^{n_l + \alpha_l + 1/\gamma}} \frac{(\tilde{s}_l + \beta_l + s_{il})^{n_l + \alpha_l}}{\Gamma(n_l + \alpha_l)} \\
&= \frac{\Gamma(n_l + \alpha_l + 1/\gamma)}{\Gamma(n_l + \alpha_l)} \frac{1}{(\tilde{s}_l + \beta_l + s_{il})^{1/\gamma}}.
\end{aligned}$$

The second equality follows from the assumption that the signals S_{1l}, \dots, S_{nl} are independent given $V_l = v_l$. The third equality follows from the assumption that V_l^γ follows a gamma distribution with parameters α_l and β_l , and $S_{il}|V_l = v_l$ an exponential distribution with parameter v_l^γ , and the well-known fact that the sum of $n_l - 1$ i.i.d. exponential variables (with mean $1/v_l^\gamma$) follows a gamma distribution with parameters v_l^γ and $n_l - 1$.

Let us now simplify somewhat the notation by suppressing n_l, z_l and θ in $x(\cdot), a(\cdot)$ and $b(\cdot)$. Since we are looking for equilibria of the form $x(p, s_{il}) = a(p) + b(p)s_{il}$, we have

$$\begin{aligned}
H(p; v_l, y) &= \Pr \left\{ \sum_{j \neq i} x(p, S_{jl}) \leq 1 - y | V_l = v_l, S_{il} = s_{il} \right\} \\
&= \Pr \left\{ \sum_{j \neq i} [a(p) + b(p)S_{jl}] \leq 1 - y | V_l = v_l, S_{il} = s_{il} \right\} \\
&= \Pr \left\{ \tilde{S}_l \geq \frac{1 - y - (n_l - 1)a(p)}{b(p)} | V_l = v_l, S_{il} = s_{il} \right\} \\
&= 1 - \Pr \left\{ \tilde{S}_l \leq \frac{1 - y - (n_l - 1)a(p)}{b(p)} | V_l = v_l, S_{il} = s_{il} \right\}
\end{aligned}$$

where the last equation follows from the fact that the functions $a(\cdot)$ and $b(\cdot)$ are such that $x(\cdot, \cdot)$ is decreasing in p and s_{il} . Letting $\tilde{s}_l = [1 - y - (n_l - 1)a(p)]/b(p)$ we have

$$H_p(p; v_l, y) = -g(\tilde{s}_l | v_l, s_{il}) \frac{\partial \tilde{s}_l}{\partial p}$$

$$H_y(p; v_l, y) = -g(\tilde{s}_l | v_l, s_{il}) \frac{\partial \tilde{s}_l}{\partial y}$$

and therefore

$$\begin{aligned}
&E \{ (V_l - p)H_p(p; V_l, y) + yH_y(p; V_l, y) | S_{il} = s_{il} \} \\
&= E \left\{ -(V_l - p)g(\tilde{s}_l | V_l, s_{il}) \frac{\partial \tilde{s}_l}{\partial p} - yg(\tilde{s}_l | V_l, s_{il}) \frac{\partial \tilde{s}_l}{\partial y} | S_{il} = s_{il} \right\} \\
&= E \left\{ -(V_l - p) | \tilde{S}_l = \tilde{s}_l, S_{il} = s_{il} \right\} g(\tilde{s}_l | s_{il}) \frac{\partial \tilde{s}_l}{\partial p} - yg(\tilde{s}_l | s_{il}) \frac{\partial \tilde{s}_l}{\partial y} \\
&= - \left(\frac{\Gamma(n_l + \alpha_l + 1/\gamma)}{\Gamma(n_l + \alpha_l)} \frac{1}{(\tilde{s}_l + \beta_l + s_{il})^{1/\gamma}} - p \right) \\
&\quad \times g(\tilde{s}_l | s_{il}) \frac{-(n_l - 1)a'(p)b(p) - b'(p)[1 - y - (n_l - 1)a(p)]}{b^2(p)} - yg(\tilde{s}_l | s_{il}) \frac{-1}{b(p)}.
\end{aligned}$$

The Euler condition (3) (including the auction-specific notation and variables) amounts to evaluating the above expression at $y = x(p, s_{il}) = a(p) + b(p)s_{il}$ and then equating the expression to zero:

$$\begin{aligned} 0 &= E \{ (V_l - p)H_p(p; V_l, y) + x(p, s_{il})H_y(p; V_l, y) | S_{il} = s_{il} \} \\ &= \left(\frac{\Gamma(n_l + \alpha_l + 1/\gamma)}{\Gamma(n_l + \alpha_l)} \frac{1}{(\beta_l + \frac{1-n_l a(p)}{b(p)})^{1/\gamma}} - p \right) \left(-\frac{(n_l - 1)a'(p)}{b(p)} - \frac{b'(p)}{b(p)} \tilde{s}_l \right) \\ &\quad - \frac{1 - (n_l - 1)a(p) - b(p)\tilde{s}_l}{b(p)} \end{aligned}$$

where we have used that $\tilde{s}_l = [1 - n_l a(p) - b(p)s_{il}]/b(p)$ so that $\tilde{s}_l + s_{il} = (1 - n_l a(p))/b(p)$. The above equality must hold for all values of s_{il} and hence for all values of \tilde{s}_l . Therefore it must be that

$$\begin{aligned} &\left\{ \left(\frac{\Gamma(n_l + \alpha_l + 1/\gamma)}{\Gamma(n_l + \alpha_l)} \frac{1}{(\beta_l + \frac{1-n_l a(p)}{b(p)})^{1/\gamma}} - p \right) \left(-\frac{(n_l - 1)a'(p)}{b(p)} \right) - \frac{1 - (n_l - 1)a(p)}{b(p)} = 0 \right. \\ &\quad \left. - \left(\frac{\Gamma(n_l + \alpha_l + 1/\gamma)}{\Gamma(n_l + \alpha_l)} \frac{1}{(\beta_l + \frac{1-n_l a(p)}{b(p)})^{1/\gamma}} - p \right) \frac{b'(p)}{b(p)} + 1 = 0 \right. \\ \Leftrightarrow &\left\{ \left(\frac{\Gamma(n_l + \alpha_l + 1/\gamma)}{\Gamma(n_l + \alpha_l)} \frac{(-b(p))^{1/\gamma}}{(-\beta_l b(p) - 1 + n_l a(p))^{1/\gamma}} - p \right) ((n_l - 1)a'(p)) + 1 - (n_l - 1)a(p) = 0 \right. \\ &\quad \left. \left(\frac{\Gamma(n_l + \alpha_l + 1/\gamma)}{\Gamma(n_l + \alpha_l)} \frac{(-b(p))^{1/\gamma}}{(-\beta_l b(p) - 1 + n_l a(p))^{1/\gamma}} - p \right) b'(p) - b(p) = 0. \right. \end{aligned} \tag{21}$$

The two equalities imply that

$$-\frac{b(p)}{b'(p)} = \frac{1 - (n_l - 1)a(p)}{(n_l - 1)a'(p)}$$

$$\Leftrightarrow -(n_l - 1)a'(p)b(p) + (n_l - 1)a(p)b'(p) = b'(p)$$

$$\Leftrightarrow (n_l - 1) \left(\frac{a(p)b'(p) - a'(p)b(p)}{b^2(p)} \right) = \frac{b'(p)}{b^2(p)}$$

$$\Leftrightarrow (n_l - 1) \frac{a(p)}{b(p)} = \frac{1}{b(p)} + C_1$$

$$\Leftrightarrow a(p) = (1 + C_1 b(p))/(n_l - 1).$$

where C_1 is the integration constant. Inserting this expression for $a(p)$ into the second equality of (21) gives

$$\begin{aligned} &\left(\frac{\Gamma(n_l + \alpha_l + 1/\gamma)}{\Gamma(n_l + \alpha_l)} \frac{(-b(p))^{1/\gamma}}{\left(-\beta_l b(p) - 1 + \frac{n_l}{n_l - 1} + \frac{n_l C_1 b(p)}{n_l - 1} \right)^{1/\gamma}} - p \right) b'(p) - b(p) = 0 \\ \Leftrightarrow &b'(p) \left[\frac{\Gamma(n_l + \alpha_l + 1/\gamma)}{\Gamma(n_l + \alpha_l)} (-b(p))^{1/\gamma} - \left(\left(\frac{C_1 n_l}{n_l - 1} - \beta_l \right) b(p) + \frac{1}{n_l - 1} \right)^{1/\gamma} p \right] \end{aligned}$$

$$- \left(\left(\frac{C_1 n_l}{n_l - 1} - \beta_l \right) b(p) + \frac{1}{n_l - 1} \right)^{1/\gamma} b(p) = 0.$$

Defining $C_2 = \frac{C_1 n_l}{n_l - 1} - \beta_l$ the last equation becomes

$$b'(p) \left[\frac{\Gamma(n_l + \alpha_l + 1/\gamma)}{\Gamma(n_l + \alpha_l)} (-b(p))^{1/\gamma} - \left(C_2 b(p) + \frac{1}{n_l - 1} \right)^{1/\gamma} p \right] - \left(C_2 b(p) + \frac{1}{n_l - 1} \right)^{1/\gamma} b(p) = 0$$

which can be rewritten as (using the transformation $u = b(p)$, so that $p(u) = b^{-1}(p)$)

$$\frac{1}{p'(u)} \left[\frac{\Gamma(n_l + \alpha_l + 1/\gamma)}{\Gamma(n_l + \alpha_l)} (-u)^{1/\gamma} - \left(C_2 u + \frac{1}{n_l - 1} \right)^{1/\gamma} p(u) \right] - \left(C_2 u + \frac{1}{n_l - 1} \right)^{1/\gamma} u = 0$$

or as

$$p'(u) \left(C_2 u + \frac{1}{n_l - 1} \right)^{1/\gamma} u + p(u) \left(C_2 u + \frac{1}{n_l - 1} \right)^{1/\gamma} = \frac{\Gamma(n_l + \alpha_l + 1/\gamma)}{\Gamma(n_l + \alpha_l)} (-u)^{1/\gamma}. \quad (22)$$

The solution of the differential equation

$$p'(u) \left(C_2 u + \frac{1}{n_l - 1} \right)^{1/\gamma} u + p(u) \left(C_2 u + \frac{1}{n_l - 1} \right)^{1/\gamma} = 0$$

is $p(u) = C_3/u$ where C_3 is an integration constant. The solution of the differential equation (22) is therefore necessarily of the form $p(u) = C(u)/u$ where $C(\cdot)$ is some function of u . Inserting this solution into (22) gives

$$\frac{C'(u)}{u} \left(C_2 u + \frac{1}{n_l - 1} \right)^{1/\gamma} u = \frac{\Gamma(n_l + \alpha_l + 1/\gamma)}{\Gamma(n_l + \alpha_l)} (-u)^{1/\gamma}$$

so that

$$C'(u) = \begin{cases} \frac{\Gamma(n_l + \alpha_l + 1/\gamma)}{\Gamma(n_l + \alpha_l)} \left[\frac{-u}{C_2 u + \frac{1}{n_l - 1}} \right]^{1/\gamma} & \text{if } C_2 \neq 0 \\ \frac{\Gamma(n_l + \alpha_l + 1/\gamma)}{\Gamma(n_l + \alpha_l)} [-(n_l - 1)u]^{1/\gamma} & \text{if } C_2 = 0 \end{cases}$$

so that

$$C(u) = \begin{cases} \frac{\Gamma(n_l + \alpha_l + 1/\gamma)}{\Gamma(n_l + \alpha_l)} \int_u^0 \left[\frac{-t}{C_2 t + \frac{1}{n_l - 1}} \right]^{1/\gamma} dt + C_4 & \text{if } C_2 \neq 0 \\ -\frac{\Gamma(n_l + \alpha_l + 1/\gamma)}{\Gamma(n_l + \alpha_l)} [(n_l - 1)]^{1/\gamma} \frac{\gamma}{1+\gamma} (-u)^{(1+\gamma)/\gamma} + C_5 & \text{if } C_2 = 0 \end{cases}$$

where C_4 and C_5 are integration constants. Since $p(u) = C(u)/u$ the solutions are therefore

$$p(u) = \begin{cases} \frac{\Gamma(n_l + \alpha_l + 1/\gamma)}{\Gamma(n_l + \alpha_l)} \frac{\int_u^0 \left[\frac{-t}{C_2 t + \frac{1}{n_l - 1}} \right]^{1/\gamma} dt}{u} + \frac{C_4}{u} & \text{if } C_2 \neq 0 \\ \frac{\Gamma(n_l + \alpha_l + 1/\gamma)}{\Gamma(n_l + \alpha_l)} [(n_l - 1)]^{1/\gamma} \frac{\gamma}{1+\gamma} (-u)^{1/\gamma} + \frac{C_5}{u} & \text{if } C_2 = 0. \end{cases} \quad (23)$$

Now recall from the main text that the functions $a(p)$ and $b(p)$ must be such that $x(p, s_{il}) = a(p) + b(p)s_{il}$ is decreasing in p and s_{il} . This implies in particular that $b(p) < 0$ and $b'(p) < 0$, which in turn implies that $p(u) \geq 0$ for all $u < 0$ and $p'(u) < 0$ for all $u < 0$. Now consider the case $C_2 = 0$, i.e. $C_1 = \frac{n_l - 1}{n_l} \beta_l$. The condition $p(u) \geq 0$ for all $u < 0$ then implies $C_5 \leq 0$, and the condition $p'(u) < 0$ for all $u < 0$ implies $C_5 \geq 0$. Thus $C_5 = 0$, which from (23) implies that

$$b(p) = - \left[\frac{\Gamma(n_l + \alpha_l)}{\Gamma(n_l + \alpha_l + 1/\gamma)} \frac{1 + \gamma}{\gamma} \right]^\gamma \frac{p^\gamma}{n_l - 1}$$

and

$$a(p) = \frac{(1 + C_1 b(p))}{n_l - 1} = \frac{1}{n_l - 1} - \frac{\beta_l}{n_l(n_l - 1)} \left[\frac{\Gamma(n_l + \beta_l)}{\Gamma(n_l + \alpha_l + 1/\gamma)} \frac{1 + \gamma}{\gamma} \right]^\gamma p^\gamma.$$

Inserting these expressions into $x(p, s_{il}) = a(p) + b(p)s_{il}$ gives (16). One can show that there does not exist a solution that verifies the conditions for the case $C_2 \neq 0$.

C Asymptotic properties of the estimator

This appendix extensively uses chapter 8 of Newey and McFadden (1994, pp. 2194-2215). The results in this appendix cannot be read independently without consulting Newey and McFadden.

In deriving the asymptotic properties of our estimator (12), we suppose, for simplicity, that $n_l = n$ for all l , and $T = 1$. The proofs are similar but more involved when the number of bidders is allowed to vary from auction to auction. The proof for $T = 1$ corresponds to what has to be done component by component for $T > 1$.

Some notations are specific to this appendix, and do not necessarily refer to the notations in the main body of the paper. We do this to facilitate the apprehension of this appendix, which heavily relies on Newey and McFadden and their notations. To avoid any possible confusion, all notations that have different meanings in the appendix and the main text, are redefined.

We first show that our estimator (12) belongs to the class of semiparametric two-step estimators considered by Newey and McFadden (1994), with the first-step estimator being a kernel estimator (section 8.3). Our estimator (12) solves the equation:

$$\frac{1}{L} \sum_{l=1}^L m(p_l^0, z_l; \theta; \hat{\gamma}(x_{.lp}, z_l); p) = 0 \quad (24)$$

where $x_{.lp} = (x_{1lp}, \dots, x_{nlp})$. As we restrict ourselves to the case $T = 1$, p corresponds in fact to p_1 in the main text. $\hat{\gamma}(\cdot, \cdot)$, evaluated at the vector $x_p = (x_{1p}, \dots, x_{np})$ and the vector z , is a vector of dimension $n + 1$, where the i -th component, with $i = 1, \dots, n$, is given by:

$$\hat{\gamma}_i(x_p, z) = \frac{1}{L} \sum_l \frac{1}{n} \sum_j 1(x_{jlp} \leq x_{ip}) K_{hz}(z - z_l)$$

and the $(n + 1)$ -th component is given by:

$$\widehat{\gamma}_{n+1}(x_p, z) = \frac{1}{L} \sum_l K_{h_z}(z - z_l)$$

where $K_{h_z}(z - z_l) = K(\frac{z-z_l}{h_z})$.

Finally, the function $m(\cdot)$ is defined by:

$$\begin{aligned} & m(p^0, z; \theta; \widehat{\gamma}(x_p, z); p) \\ = & \left\{ \begin{array}{l} (n-1)(E[V|S_1 = F_{S|Z}^{-1}(1 - \frac{\widehat{\gamma}_1(x_p, z)}{\gamma_{n+1}^0(x_p, z)}|z; \theta), \dots, S_n = F_{S|Z}^{-1}(1 - \frac{\widehat{\gamma}_n(x_p, z)}{\gamma_{n+1}^0(x_p, z)}|z; \theta), z] - p) \\ -(p - p^0) \end{array} \right\} \\ & \times 1(p^0 \leq p). \end{aligned}$$

Note that this function $m(\cdot)$ is not exactly the function (11) defined in the main text.²⁰

Thus, the equation that defines our estimator, (24), can be viewed as a special case of equations (8.1) and (8.7) in Newey and McFadden. To apply their theorems 8.11, 8.12, and 8.13, we introduce the following three additional functions.

First of all, $\nu(p^0, z, x_p; \theta^0; p)$ which is a vector of dimension $(n + 1)$, where the i -th component, with $i = 1, \dots, n$, is given by

$$\begin{aligned} & \nu_i(p^0, z, x_p; \theta^0; p) \\ = & - \left\{ \frac{\partial E[V|S_1, \dots, S_n, Z]}{\partial S_i} (F_{S|Z}^{-1}(1 - \frac{\gamma_1^0(x_p, z)}{\gamma_{n+1}^0(x_p, z)}|z; \theta^0), \dots, F_{S|Z}^{-1}(1 - \frac{\gamma_n^0(x_p, z)}{\gamma_{n+1}^0(x_p, z)}|z; \theta^0), z) \right\} \\ & \times (n-1) F_{S|Z}^{-1'}(1 - \frac{\gamma_i^0(x_p, z)}{\gamma_{n+1}^0(x_p, z)}|z; \theta^0) \frac{1(p^0 \leq p)}{\gamma_{n+1}^0(x_p, z)} g^0(x_p, z) \end{aligned}$$

and where the $(n + 1)$ -th component is given by

$$\begin{aligned} & \nu_{(n+1)}(p^0, z, x_p; \theta^0; p) \\ = & \sum_{i=1}^n \left\{ \frac{\partial E[V|S_1, \dots, S_n, Z]}{\partial S_i} (F_{S|Z}^{-1}(1 - \frac{\gamma_1^0(x_p, z)}{\gamma_{n+1}^0(x_p, z)}|z; \theta^0), \dots, F_{S|Z}^{-1}(1 - \frac{\gamma_n^0(x_p, z)}{\gamma_{n+1}^0(x_p, z)}|z; \theta^0), z) \right\} \\ & \times (n-1) F_{S|Z}^{-1'}(1 - \frac{\gamma_i^0(x_p, z)}{\gamma_{n+1}^0(x_p, z)}|z; \theta^0) \frac{\gamma_i^0(x_p, z) 1(p^0 \leq p)}{(\gamma_{n+1}^0(x_p, z))^2} g^0(x_p, z). \end{aligned}$$

Second, $\mu(z, x_p; \theta^0; p)$ which is a vector of dimension n where the i -th term is given by:

$$\mu_i(z, x_p; \theta^0; p) = \int \nu_i(p^0, z, (y_{1p}, \dots, y_{(i-1)p}, x_{ip}, y_{(i+1)p}, \dots, y_{np}); \theta^0; p) dy_{1p} \dots dy_{(i-1)p} dy_{(i+1)p} \dots dy_{np}$$

²⁰Note also that the function $G(\cdot)$ in the main text corresponds in the appendix to ratios of the form $\frac{\widehat{\gamma}_i(x_p, z)}{\gamma_{1+n}^0(x_p, z)}$.

Third, $\lambda(z; \theta^0; p)$ which is a scalar given by

$$\lambda(z; \theta^0; p) = \int \nu_{(n+1)}(p^0, z, (y_{1p}, \dots, y_{np}); \theta^0; p) dy_{1p} \dots dy_{np}.$$

In our context, under their hypotheses, Theorem 8.12 of Newey and McFadden can be written as:

Theorem 1

$$\sqrt{L}(\hat{\theta} - \theta^0) \rightarrow N(0, (M'_\theta M_\theta)^{-1} M'_\theta \Omega M_\theta (M'_\theta M_\theta)^{-1})$$

where

$$M_\theta = E[\nabla_\theta m(P^0, Z; \theta^0; \gamma^0(X_p, Z); p)]$$

and

$$\Omega = \text{Var}[m(P^0, Z; \theta^0; \gamma^0(X_p, Z); p) + \delta(Z, X_p; \theta^0; p)]$$

and where $\delta(z, x_p; \theta^0; p)$ is a scalar given by:

$$\delta(z, x_p; \theta^0; p) = \frac{1}{n} \sum_{j=1}^n \sum_{i=1}^n \int_{x_{jp}}^{+\infty} \mu_i(z, y_p; \theta^0; p) dy_{ip} + \lambda(z; \theta^0; p).$$

Proof. Newey and McFadden have introduced the function δ in their theorems 8.11 and 8.12. What has to be proved is that this function δ takes in our context the specific form given in our theorem. For this we follow, step by step, the reasoning of Newey and McFadden (pp. 2207-2208).

The first step consists in linearizing the function $m(p^0, z; \theta^0; \gamma(x_p, z); p)$ around the true value $\gamma^0(x_p, z)$. This development allows us to find $M(p^0, z; \theta^0; \gamma(x_p, z); p)$, the linearization of $m(\cdot)$. Here $M(\cdot)$ is a function given by :

$$\begin{aligned} & M(p^0, z; \theta^0; \gamma(x_p, z); p) \\ = & - \sum_{i=1}^n \left\{ \frac{\partial E[V|S_1, \dots, S_n, Z]}{\partial S_i} (F_{S|Z}^{-1}(1 - \frac{\gamma_1^0(x_p, z)}{\gamma_{n+1}^0(x_p, z)}|z; \theta^0), \dots, F_{S|Z}^{-1}(1 - \frac{\gamma_n^0(x_p, z)}{\gamma_{n+1}^0(x_p, z)}|z; \theta^0), z) \right\} \\ & \times (n-1) F_{S|Z}^{-1}(1 - \frac{\gamma_i^0(x_p, z)}{\gamma_{n+1}^0(x_p, z)}|z; \theta^0) \frac{1(p^0 \leq p)}{\gamma_{n+1}^0(x_p, z)} \gamma_i(x_p, z) \\ & + \sum_{i=1}^n \left\{ \frac{\partial E[V|S_1, \dots, S_n, Z]}{\partial S_i} (F_{S|Z}^{-1}(1 - \frac{\gamma_1^0(x_p, z)}{\gamma_{n+1}^0(x_p, z)}|z; \theta^0), \dots, F_{S|Z}^{-1}(1 - \frac{\gamma_n^0(x_p, z)}{\gamma_{n+1}^0(x_p, z)}|z; \theta^0), z) \right\} \\ & \times (n-1) F_{S|Z}^{-1}(1 - \frac{\gamma_i^0(x_p, z)}{\gamma_{n+1}^0(x_p, z)}|z; \theta^0) \frac{\gamma_i^0(x_p, z) 1(p^0 \leq p)}{(\gamma_{n+1}^0(x_p, z))^2} \gamma_{n+1}(x_p, z). \end{aligned}$$

The next step is to calculate the function $\nu(p^0, z; \theta^0; p)$ defined by

$$\int_z \int_{x_p} M(p^0, z; \theta^0; \gamma(x_p, z); p) g^0(x_p, z) dx_p dz = \int_z \int_{x_p} \nu(p^0, z, x_p; \theta^0; p) \gamma(x_p, z) dx_p dz$$

where $g^0(x_p, z)$ is the joint distribution of x_{1p}, \dots, x_{np} and z . This is equation 8.12 of Newey and McFadden. The function $\nu(\cdot)$ that solves this equation is the function $\nu(\cdot)$ defined previously.

Finally, we have to calculate the function $\delta(\cdot)$. For this purpose, we rely on equation 8.14 of Newey and McFadden, and define $\delta(\cdot)$ as the solution of the following equation:

$$\int_z \int_{x_p} M(p^0, z; \theta^0; \widehat{\gamma}(x_p, z) - \gamma^0(x_p, z); p) g^0(x_p, z) dx_p dz = \int_z \int_{x_p} \delta(z, x_p; \theta^0; p) \widehat{g}(x_p, z) dx_p dz \quad (25)$$

where, as in Newey and McFadden, the integral of a function $a(z, x_p)$ over $d\widehat{g}$ is equal to $\frac{1}{L} \sum_{l=1}^L \int a(z, x_{lp}) K_{h_z}(z - z_l) dz$. Given the functions $\mu(\cdot)$ and $\lambda(\cdot)$ introduced previously, we obtain:

$$\begin{aligned} \int_z \int_{x_p} \nu(p^0, z, x_p; \theta^0; p) \widehat{\gamma}(x_p, z) dx_p dz &= \frac{1}{L} \sum_l \frac{1}{n} \sum_j \sum_i \int dz \int_{x_{jlp}}^{+\infty} \mu_i(z, x_p; \theta^0; p) K_{h_z}(z - z_l) dx_{ip} \\ &+ \frac{1}{L} \sum_l \int \lambda(z; \theta^0; p) K_{h_z}(z - z_l) dz \end{aligned}$$

and

$$\begin{aligned} \int_z \int_{x_p} \nu(p^0, z, x_p; \theta^0; p) \gamma^0(x_p, z) dx_p dz &= \int_z \int_{x_p} \left\{ \begin{array}{l} \sum_i \nu_i(p^0, z, x_p; \theta^0; p) \gamma_i^0(x_p, z) \\ + \nu_{(n+1)}(p^0, z, x_p; \theta^0; p) \gamma_{n+1}^0(x_p, z) \end{array} \right\} \\ &= 0. \end{aligned}$$

The function δ that solves (25) is thus defined by

$$\delta(z, x_p; \theta^0; p) = \frac{1}{n} \sum_{j=1}^n \sum_{i=1}^n \int_{x_{jp}}^{+\infty} \mu_i(z, y_p; \theta^0; p) dy_{ip} + \lambda(z; \theta^0; p)$$

as stated in our theorem.

This allows us to obtain the analogue of Newey and McFadden's equation 8.15, that is:

$$\begin{aligned} &\int_z \int_{x_p} \delta(z, x_p; \theta^0; p) \widehat{g}(x_p, z) dx_p dz - \int_z \int_{x_p} \delta(z, x_p; \theta^0; p) \widetilde{g}(x_p, z) dx_p dz \\ &= \frac{1}{L} \sum_l \frac{1}{n} \sum_j \sum_i \int_{x_{jlp}}^{+\infty} \left\{ \int \mu_i(z, x_p; \theta^0; p) K_{h_z}(z - z_l) dz - \mu_i(z_l, x_p; \theta^0; p) \right\} dx_{ip} \\ &+ \frac{1}{L} \sum_l \left\{ \int \lambda(z; \theta^0; p) K_{h_z}(z - z_l) dz - \lambda(z_l; \theta^0; p) \right\} \end{aligned}$$

where $\widetilde{g}(x_p, z)$ is the empirical distribution of (x_p, z) , and $\widehat{g}(x_p, z)$, defined previously, corresponds to the distribution of (x_p, z) where the empirical distribution of z has been replaced by a smoothed version $\frac{1}{L} \sum_l K_{h_z}(z - z_l)$, exactly as in Newey and McFadden. This equation is important to verify the condition (iv) of theorem 1 (page 2196) in Newey and McFadden, which is a necessary condition to apply theorems 8.11 and 8.12.

■

It has been shown that our problem fits exactly in the framework of Newey and McFadden, and we have defined in our context all the functions introduced in their general case.

Newey and McFadden also study the estimation of the asymptotic variance-covariance matrix. By introducing the three following functions:

$$\widehat{M}_\theta = \frac{1}{L} \sum_{l=1}^L \nabla_\theta m(p_l^0, z_l; \widehat{\theta}; \widehat{\gamma}(x_{lp}, z_l); p)$$

$$\widehat{\lambda}(z; \widehat{\theta}; \widehat{\gamma}; p) = \frac{1}{L} \sum_l \sum_{i=1}^n \left\{ \frac{\partial E[V|S_1, \dots, S_n, Z]}{\partial S_i} (F_{S|Z}^{-1}(1 - \frac{\widehat{\gamma}_1(x_{lp}, z_l)}{\widehat{\gamma}_{n+1}(x_{lp}, z_l)} | z_l; \widehat{\theta}), \dots, F_{S|Z}^{-1}(1 - \frac{\widehat{\gamma}_n(x_{lp}, z_l)}{\widehat{\gamma}_{n+1}(x_{lp}, z_l)} | z_l; \widehat{\theta}), z_l) \right\}$$

$$\times (n-1) F_{S|Z}^{-1'}(1 - \frac{\widehat{\gamma}_i(x_{lp}, z_l)}{\widehat{\gamma}_{n+1}(x_{lp}, z_l)} | z_l; \widehat{\theta}) \frac{\widehat{\gamma}_i(x_{lp}, z_l) 1(p_l^0 \leq p)}{(\widehat{\gamma}_{n+1}(x_{lp}, z_l))^2} K_{h_Z}(z - z_l)$$

$$\widehat{\rho}_{ij}(z, x_p; \widehat{\theta}; \widehat{\gamma}; p) = \frac{1}{L} \sum_l \left\{ \frac{\partial E[V|S_1, \dots, S_n, Z]}{\partial S_i} (F_{S|Z}^{-1}(1 - \frac{\widehat{\gamma}_1(x_{lp}, z_l)}{\widehat{\gamma}_{n+1}(x_{lp}, z_l)} | z_l; \widehat{\theta}), \dots, F_{S|Z}^{-1}(1 - \frac{\widehat{\gamma}_n(x_{lp}, z_l)}{\widehat{\gamma}_{n+1}(x_{lp}, z_l)} | z_l; \widehat{\theta}), z_l) \right\}$$

$$\times (n-1) F_{S|Z}^{-1'}(1 - \frac{\widehat{\gamma}_i(x_{lp}, z_l)}{\widehat{\gamma}_{n+1}(x_{lp}, z_l)} | z_l; \widehat{\theta}) \frac{1(p_l^0 \leq p)}{\widehat{\gamma}_{n+1}(x_{lp}, z_l)} 1(x_{jp} \leq x_{ilp}) K_{h_Z}(z - z_l)$$

we obtain, under the hypotheses of Newey and McFadden, the analogue of their theorem 8.13:

Theorem 2

$$(\widehat{M}'_\theta \widehat{M}_\theta)^{-1} \widehat{M}'_\theta \widehat{\Omega} \widehat{M}_\theta (\widehat{M}'_\theta \widehat{M}_\theta)^{-1} \rightarrow (M'_\theta M_\theta)^{-1} M'_\theta \Omega M_\theta (M'_\theta M_\theta)^{-1}$$

where

$$\widehat{\Omega} = \frac{1}{L} \sum_{l=1}^L [m(p_l^0, z_l; \widehat{\theta}; \widehat{\gamma}(x_{lp}, z_l); p) + \widehat{\delta}(z_l, x_{lp}; \widehat{\theta}; \widehat{\gamma}; p)] [m(p_l^0, z_l; \widehat{\theta}; \widehat{\gamma}(x_{lp}, z_l); p) + \widehat{\delta}(z_l, x_{lp}; \widehat{\theta}; \widehat{\gamma}; p)]'$$

and

$$\widehat{\delta}(z, x_p; \widehat{\theta}; \widehat{\gamma}; p) = \frac{1}{n} \sum_{j=1}^n \sum_{i=1}^n \widehat{\rho}_{ij}(z, x_p; \widehat{\theta}; \widehat{\gamma}; p) + \widehat{\lambda}(z; \widehat{\theta}; \widehat{\gamma}; p).$$

Proof. The function $\widehat{\delta}(z, x_p; \theta; \gamma; p)$ itself has to be estimated (whereas in Newey and McFadden the function $\delta(\cdot)$ is assumed to be known; see their equation 8.17). This is not a problem as shown by the following decomposition:

$$\| \widehat{\delta}(z, x_p; \widehat{\theta}; \widehat{\gamma}; p) - \delta(z, x_p; \theta^0; \gamma^0; p) \| \leq \| \widehat{\delta}(z, x_p; \widehat{\theta}; \widehat{\gamma}; p) - \widehat{\delta}(z, x_p; \theta^0; \gamma^0; p) \| + \| \widehat{\delta}(z, x_p; \theta^0; \gamma^0; p) - \delta(z, x_p; \theta^0; \gamma^0; p) \|.$$

The first term of the decomposition is of the type considered by Newey and McFadden, and converges to zero under their hypotheses. The second term is a standard one, and converges to zero as expectations are estimated by empirical means.

■

Finally, for the previous asymptotic results to be valid, the distribution functions $F_{V|Z}(\cdot)$ and $F_{S|V,Z}(\cdot)$ should be chosen such that the conditions of our theorems 1 and 2 (i.e. the hypotheses of Newey and McFaddens' theorems)

are verified. It can be shown that for our particular choices of the distribution functions (see the specifications 13 and 14), this is indeed the case. The calculations that these verifications entail are lengthy and fastidious and are therefore omitted. They are based on first -and second order developments, and can be obtained from the authors upon request.

Table A1. The auctions

| Type of auction | Auction date | Line auctioned | Settlement date | Maturity date |
|-----------------|--------------|----------------|-----------------|---------------|
| OAT | 5 Jan 95 | 7.5% 2005 | 25 Jan 95 | 25 Apr 05 |
| OAT | 5 Jan 95 | 6% 2025 | 25 Jan 95 | 25 Oct 25 |
| OAT | 2 Feb 95 | 7.5% 2005 | 27 Feb 95 | 25 Apr 05 |
| OAT | 2 Feb 95 | 8.5% 2008 | 27 Feb 95 | 25 Oct 08 |
| OAT | 2 Feb 95 | 6% 2025 | 27 Feb 95 | 25 Oct 25 |
| OAT | 2 Mar 95 | 7.5% 2005 | 27 Mar 95 | 25 Apr 95 |
| OAT | 2 Mar 95 | 8.5% 2008 | 27 Mar 95 | 25 Oct 08 |
| OAT | 6 Apr 95 | 7.75% 2005 | 25 Apr 95 | 25 Apr 05 |
| OAT | 4 May 95 | 7.75% 2005 | 26 May 95 | 25 Apr 05 |
| OAT | 4 May 95 | 8.5% 2008 | 26 May 95 | 25 Oct 08 |
| OAT | 1 Jun 95 | 8.5% 2002 | 26 Jun 95 | 25 Nov 02 |
| OAT | 1 Jun 95 | 7.75% 2005 | 26 Jun 95 | 25 Apr 05 |
| OAT | 1 Jun 95 | 6% 2025 | 26 Jun 95 | 25 Apr 25 |
| OAT | 6 Jul 95 | 7.75% 2005 | 25 Jul 95 | 25 Oct 05 |
| OAT | 6 Jul 95 | 8.5% 2008 | 25 Jul 95 | 25 Oct 08 |
| OAT | 6 Jul 95 | 6% 2025 | 25 Jul 95 | 25 Oct 25 |
| OAT | 3 Aug 95 | 8.5% 2002 | 25 Aug 95 | 25 Nov 02 |
| OAT | 3 Aug 95 | 7.75% 2005 | 25 Aug 95 | 25 Oct 05 |
| OAT | 3 Aug 95 | 8.5% 2008 | 25 Aug 95 | 25 Oct 08 |
| OAT | 7 Sep 95 | 7.75% 2005 | 25 Sep 95 | 25 Oct 05 |
| OAT | 7 Sep 95 | 6% 2025 | 25 Sep 95 | 25 Oct 25 |
| OAT | 5 Oct 95 | 7.25% 2006 | 25 Oct 95 | 25 Apr 06 |
| OAT | 2 Nov 95 | 7.25% 2006 | 27 Nov 95 | 25 Apr 06 |
| OAT | 2 Nov 95 | 6% 2025 | 27 Nov 95 | 25 Oct 25 |
| OAT | 7 Dec 95 | 7.25% 2006 | 26 Dec 95 | 25 Apr 06 |
| BTAN | 19 Jan 95 | 7.75% 2000 | 13 Feb 95 | 12 Apr 00 |
| BTAN | 16 Feb 95 | 6.5% 1996 | 6 Mar 95 | 12 Oct 96 |
| BTAN | 16 Feb 95 | 7.75% 2000 | 13 Mar 95 | 12 Apr 00 |
| BTAN | 16 Mar 95 | 7.25% 1997 | 5 Apr 95 | 12 Aug 97 |
| BTAN | 20 Apr 95 | 7.25% 1997 | 5 May 95 | 12 Aug 97 |
| BTAN | 20 Apr 95 | 7.75% 2000 | 12 May 95 | 12 Apr 00 |
| BTAN | 18 May 95 | 7.25% 1997 | 6 Jun 95 | 12 Aug 97 |
| BTAN | 18 May 95 | 7.75% 2000 | 12 Jun 95 | 12 Apr 00 |
| BTAN | 15 Jun 95 | 7.25% 1997 | 5 Jul 95 | 12 Aug 97 |
| BTAN | 15 Jun 95 | 7.75% 2000 | 12 Jul 95 | 12 Apr 00 |
| BTAN | 20 Jul 95 | 7% 2000 | 11 Aug 95 | 12 Oct 00 |
| BTAN | 17 Aug 95 | 7.25% 1997 | 5 Sep 95 | 12 Aug 97 |
| BTAN | 17 Aug 95 | 7% 2000 | 12 Sep 95 | 12 Oct 00 |
| BTAN | 21 Sep 95 | 7.25% 1997 | 5 Oct 95 | 12 Aug 97 |
| BTAN | 21 Sep 95 | 7% 2000 | 12 Oct 95 | 12 Oct 00 |
| BTAN | 19 Oct 95 | 7.25% 1997 | 6 Nov 95 | 12 Aug 97 |
| BTAN | 19 Oct 95 | 7% 2000 | 13 Nov 95 | 12 Oct 00 |
| BTAN | 16 Nov 95 | 5.75% 1998 | 5 Dec 95 | 12 Mar 98 |
| BTAN | 21 Dec 95 | 5.75% 1998 | 5 Jan 96 | 12 Mar 98 |
| BTAN | 21 Dec 95 | 7% 2000 | 12 Jan 96 | 12 Oct 00 |

Table A2. Expected value of V , stop-out price, and secondary market price

| $E[V_i S_{1l}, \dots, S_{n_l}, Z_l]$ | Stop-out price | $E[V_i Z_l]$ | Secondary market Price |
|--------------------------------------|----------------|--------------|------------------------|
| 96.28 | 93.96 | 96.40 | 94.17 |
| 71.75 | 70.88 | 72.09 | 71.33 |
| 97.75 | 95.72 | 97.80 | 95.86 |
| 104.37 | 102.26 | 104.40 | 102.39 |
| 75.15 | 72.94 | 75.41 | 73.35 |
| 98.72 | 96.56 | 98.77 | 96.91 |
| 105.65 | 103.28 | 105.67 | 103.45 |
| 101.98 | 99.68 | 101.99 | 99.65 |
| 102.36 | 99.74 | 102.39 | 99.75 |
| 107.63 | 104.70 | 107.65 | 104.80 |
| 111.79 | 106.74 | 111.80 | 106.80 |
| 105.84 | 101.76 | 105.82 | 102.37 |
| 80.70 | 77.46 | 80.55 | 78.07 |
| 103.97 | 101.20 | 103.94 | 101.21 |
| 110.49 | 106.46 | 110.51 | 106.25 |
| 79.27 | 76.18 | 79.22 | 76.80 |
| 115.56 | 107.96 | 115.57 | 107.91 |
| 107.03 | 102.80 | 106.99 | 103.00 |
| 118.80 | 108.16 | 118.80 | 108.50 |
| 106.71 | 102.72 | 106.68 | 102.85 |
| 79.92 | 77.22 | 79.84 | 77.50 |
| 99.87 | 97.78 | 99.88 | 98.16 |
| 102.01 | 99.62 | 102.02 | 99.75 |
| 80.20 | 77.80 | 80.05 | 77.80 |
| 108.32 | 102.60 | 108.27 | 102.80 |
| 101.07 | 99.41 | 101.10 | 99.50 |
| 97.69 | 99.21 | 97.81 | 99.01 |
| 101.87 | 100.19 | 101.86 | 100.14 |
| 99.90 | 99.21 | 99.99 | 99.24 |
| 101.02 | 99.97 | 101.02 | 99.95 |
| 102.83 | 100.90 | 102.81 | 100.88 |
| 104.03 | 101.47 | 104.02 | 101.54 |
| 106.77 | 103.27 | 106.74 | 103.21 |
| 103.25 | 101.30 | 103.23 | 101.24 |
| 106.08 | 102.80 | 106.04 | 102.87 |
| 101.59 | 99.90 | 101.61 | 100.06 |
| 104.90 | 101.96 | 104.85 | 102.02 |
| 102.98 | 100.78 | 102.95 | 100.75 |
| 105.29 | 102.13 | 105.24 | 102.19 |
| 104.76 | 101.47 | 104.72 | 101.53 |
| 103.94 | 101.61 | 103.90 | 101.60 |
| 102.56 | 100.60 | 102.54 | 100.57 |
| 99.58 | 100.36 | 99.31 | 100.42 |
| 105.56 | 100.73 | 105.29 | 100.78 |
| 115.78 | 103.91 | 115.72 | 104.08 |