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Knut Wicksell and the Challenges Faced by Capital Theory

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Abstract

The review of Wicksell's contributions to capital theory focuses on stationary states in four distinct models he used to account for the time dimension of production in market economies. We now better understand the true challenges because of two developments : a full theory of intertemporal general equilibria, the methodology for comparative analysis explained by Hicks. In Wicksell's equilibria, the real interest rate turns out to be equal to the marginal productivity of the volume of social capital, a concept that he could not master, hence avoided. Challenges remain, particularly to best account for complementarities and substitutions through time.

Résumé

Les contributions de Wicksell à la théorie pure du capital ont fait l'objet de ses deux livres publiés en 1893 et 1901. L'édition définitive du second, parue en 1928, contient une importante annexe formalisant le « problème du Dr. Akerman ». Ce sont au total les traitements de trois modèles que Wicksell a laissés, auquel s'ajoute l'esquisse d'un quatrième modèle repris ultérieurement par G.-H. Bousquet (1936), J. Hicks (1939) et M. Allais (1947). Motivé surtout par l'analyse des processus détournés de production qui exigent l'emploi du capital, Wicksell a cherché à caractériser et à comparer les états stationnaires de ses modèles. Nous pouvons aujourd'hui pousser plus loin l'analyse et résoudre des difficultés qui l'avaient arrêté, en particulier pour le traitement de la relation entre le taux réel d'intérêt et la productivité marginale du capital social. Nous le pouvons grâce aux acquis modernes de la théorie de l'équilibre général et grâce à la méthodologie que proposa Hicks pour l'analyse comparative, précisément en vue de traiter le quatrième modèle de Wicksell. Les trois autres modèles formalisent d'importantes complémentarités intertemporelles qui mériteraient d'être mieux prises en compte alors que les substituabilités intertemporelles dominent trop les théories modernes.

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1. Introduction

The reader should realize that I find myself here in a rather exceptional and somewhat emotional situation. I have to comment about the contributions of Knut Wicksell to the pure theory of capital. Hundred years ago he was just publishing the first edition of volume I of his *Lectures on Political Economy*, the volume in which he was explaining capital theory and its consistency with the general theory of value, a subject which had already been the main theme of his first book *Value, Capital and Rent,* published in 1893. Central in his explanation was the articulation between the study of stationary states, the focus of capital theory, and the intertemporal interpretation of the theory of value.

Exactly fifty years ago I was trying to clarify further precisely the same points for what would become my article Malinvaud (1953), probably my only contribution to deep theory. While addressing this symposium I cannot refrain from recollecting the difficulties I was then facing. Few commentators ever had the fortune to speak with a similar historical perspective in the background.

A very abridged summary of Wicksell's contributions to capital theory may run approximately as follows. Wicksell accepted the essentials of Böhm-Bawerk's theory of interest and understood they were fully consistent with Walras' theory of general equilibrium duly extended. His main concern was then to investigate the properties following from one of Böhm-Bawerk's theses, according to which more roundabout techniques of production were more productive.

In order to look in retrospect at this work after a century, we must start from some vision of the present state of pure capital theory. For that purpose we may say that the theory in question is made of two parts : first, a general abstract model which unfortunately has very little predictive power, second, a large set of more specific models, each one relying on restrictive hypotheses, particularly with respect to aggregation. When we must apply capital theory we have to draw from the set of specific models a particular one, which we may claim to be appropriate for the question at issue. The real challenge is to know whether the selection is adequate.

Section 2 here will argue that this division is implicit in Wicksell's writings and leads to the same broad methodology which we are now using. In particular capital theory is recognized to be made of two sides, the consumption side, namely the ultimate choice between present and future goods, and the production side which, loosely speaking, results in a higher or lower marginal productivity of capital. Wicksell early realized that the main challenges of capital theory concerned the production side, because it is where matters are really complicated. This is why most of this paper will also be devoted to survey the analysis developed by Wicksell from four specific production models in turn. Section 3 will concern the 1893 model of *Value, Capital and Rent*, section 4 the « point input-point output model » as presented in the main body of the *Lectures on Political Economy*. Section 5 will sketch a kind of more general model also presented in the *Lectures* but mainly investigated later by other mathematical economists, particularly by John Hicks. Section 6 will deal with the 1923 *Analysis of Dr. Akerman's problem,* published in the English edition of the *Lectures*. The last section will discuss interpretations given to Wicksell's theory of capital in the 1960s and 1970s, at the time of the controversy between the two Cambridge[2].

[2] Exact references will be given to pages of the English editions, denoted simply as *Value* for the 1954 translation of the 1893 book and *Lectures* for the 1934 translation of volume I of the third edition of the

Reading again after so many decades Wicksell's original writings leaves us impressed by the astonishing value of his scientific insights. We can only subscribe to what P. Samuelson (1987) wrote when he concluded : « Wicksell's economics, because of its eclecticism and generality, adapts well to the present post-neoclassical age. As with Cournot, his writings speak eloquently to readers of a later century ».

2. An intertemporal general equilibrium theory focusing on stationary states

Wicksell was very brief on the presentation of his intertemporal general equilibrium theory, on the ground that it could be found in Böhm-Bawerk's *Positive Theory of Capital* (1889). After his first part explaining the atemporal « New Theory of Value », Wicksell writes in 1893 : « How does interest arise, and in particular, how can consumable goods bear interest ?… I should like to let Böhm-Bawerk speak on this question… If we cannot agree with all his conclusions, yet we must gratefully acknowledge that scarcely any other author has penetrated so deeply as he into the real nature of the matter… The simple formula in which Böhm-Bawerk wishes to comprehend all phenomena in the realm of capital interest… runs as follows : *Interest is an agio which comes into being when present and future goods are exchanged.*… The clarifying element, newly added, lies in the word *exchange* » (*Value*, p. 106-107).

In order to develop his theory Böhm-Bawerk had first considered the pure exchange economy. The demands of each agent involve « a subjective rate of interest at which he is prepared, given his preferences over time and his (expected) income over time, to exchange subjectively certain prospects of the commodities available in the future for the same amount of commodities available in the present. [Agents] also typically exhibit positive time preference : commodities available in the present are typically evaluated at higher prices than subjectively certain prospects of the same commodities available in the future » (quotation from K. Hennings, 1987). Böhm-Bawerk had then extended his model in order to include production and to find other determinants of the rate of interest.

Wicksell had two motivations in his research on capital theory. The first one directly followed from his difficulties with a number of arguments put forward by Böhm-Bawerk in order to explain the various reasons why interest rates were positive. These difficulties are listed, right after the main high appreciation quoted above, in pages 108 to 115 of *Value*, where we can read sentences like the following : « In my opinion, Böhm-Bawerk must be blamed for having mixed up the two questions of the origin of interest and the origin of interest-bearing capital itself - in his criticism of the older theories of interest as well as in his own positive presentation – instead of separating them in a truly scientific manner » (p. 113). This position of Wicksell was not revised in subsequent years since we read in the final edition of the *Lectures* : « In spite of his brilliant style, Böhm-Bawerk's exposition is marred by a rather excessive diffuseness… On the other hand, in my opinion, his logical analysis of the subject was, in one important respect, not carried as far as would be desirable from an expository point of view. I propose, therefore, to present here Böhm-Bawerk's principal ideas in an abridged and, if possible, clearer and more comprehensible form » (p. 147).

Vorlesungen published in 1928 after the death of the author, a translation which was supplemented by that of two of Wicksell's longer articles, in particular the one discussing Dr. Akerman's problem (1923).

The second motivation of Wicksell was to contribute to a synthetic theory of production and distribution, meant to serve the same purpose as for instance the theory of David Ricardo. Already in 1893 the full title of the second part of his book was not only « The New Theory of Capital » but also « and its Relations to the Theory of Wages, Ground-Rents and Values of Goods ». In the *Lectures* the second part discussing capital theory has simply the title « The Theory of Production and Distribution ». The bibliography, presented right under this title, contains the following two sentences : « There still exists no exhaustive presentation of this subject on modern lines » ; « Böhm-Bawerk…, whose work… is the chief source for the modern theory of capital, did not concern himself with the synthetic treatment of the problem of production and distribution as a whole » (p. 101).

These quotations reveal the high ambitions Wicksell was entertaining. But his work on simple mathematical specifications had persuaded him that the problems to be solved were really challenging, and probably also that intuition might be misleading because of the spontaneous attraction of too easy transpositions. In order to discuss Böhm-Bawerk various reasons for positive interest rates and in order to build synthetic models of production and distribution, Wicksell needed a principle which would circumvent quite a few of these difficulties. Focusing on stationary states provided the required principle.

The following quotation from the Preface of *Value* (p. 21-22) is worth reading in full. «As I see it, these two questions – the question of the origin of capital interest and that of the origin of interest-bearing capital itself – however closely they are related, must in theory be first of all treated separately, just as, for instance, the theory of exchange is separated from the theory of production, though in reality exchange and production are almost always dependent on each other. If one does this, or what comes to about the same : if one takes as a fundamental – and simplest – hypothesis the *stationary* economy in which capital and the other economic factors can be thought of as an approximately unalterable sum, then some of the most important of Böhm-Bawerk's objections to the older theories lose their significance, and the theory of productivity as well as the Use theory can then be applied to the investigation of the actual phenomena of interest quite as appropriately as his own – and in some respects even more so ».

Overall, we can see that Wicksell would feel quite at home in the framework within which our present pure theory of capital finds its place. This framework contains the competitive intertemporal equilibrium, the sequence of temporary equilibria and the particular case of stationary equilibria. The intertemporal equilibrium provides our theory of value, just as Wicksell presents the theory of capital as fully consistent with the theory of value. We saw the central place recognized in Wicksell's opus to the study of stationary equilibria. The concept of a temporary equilibrium is not explained in the two books here analysed. But it plays a part in *Interest and Prices,* published by Wicksell in 1898. Indeed, J. Hicks in *Value and Capital* (1939) identifies Wicksell together with Walras and Pareto as one of the three writers from whom he took inspiration for building his theory of the general temporary equilibrium (p. 2). He also recognizes the influence of Wicksell on his discussion of the imperfect stability of this equilibrium (p. 251-3).

Volume I of the *Lectures* itself devotes a brief twelve-page third part to Capital accumulation, which is in essence just the result of the sequence of temporary equilibria. Wicksell writes at the beginning : «A rational theory of saving is... necessary before we can clearly understand the conditions of a stationary society, with a constant supply of capital ; and still more, of course, before we can understand and foresee the gradual changes in the

amount of social capital » (p. 207). But he had warned us in his bibliography : «The literature on this subject is very meager ». Ending this part he writes : «What has been said [here] may suffice to indicate, rather than solve, the many problems associated with the question of capital accumulation which has been so little investigated » (p. 217).

3. How roundabout should production be ?

A good entry for the study of Wicksell's models of production is provided by the following sentence which we read in *Value* just at the beginning of the presentation of Wicksell's own theory. « In the last analysis [the role of capital in production consists] simply and solely in making possible the introduction of a longer period of time between the beginning and the conclusion of the process of production of the commodity concerned and consequently the adoption of a more round-about method of production than would be possible if production were less strong in capital or totally devoid of capital » (p. 115). As we shall see in the following sections, the subsequent development of Wicksell's research on capital theory led him to somewhat withdraw from such a categorical formulation. But the model behind it is worth remembering.

The intention here is not to give a complete survey of the contribution of the 1893 book to capital theory. In particular for the sake of brevity, an important part of the book will be fully neglected, the one in which the author examines the role of rents and of «rent-goods », the highly durable goods which are either provided by nature or the result of past long-term investments. From the full title of the book, from the section on pages 146-153 and from references given to it by Wicksell in subsequent writings, it is clear that he thought the analysis given there was significant. We shall also neglect the discussion in the book of other environments than perfect competition. Finally, concerning the case of one primary factor only (labour) and perfect competition, attention will be limited to the mathematical model : this means ignoring the care with which the author explained why he thought this model was providing a valuable first approximation for the analysis of the role of circulating capital in real economies[3].

The model

The model, described in modern terms and with a modern notation, has two commodities : a homogeneous labour available in quantity L, paid a real wage w, and a produced good serving as numeraire. The technology is made of a one-dimensional family of techniques indexed by θ, the length of their period of production. The technique θ has constant returns to scale and, for producing an output equal to 1, requires a constant flow of labour services during the whole period of production θ. Thus, using the appropriate technique at time t in a stationary state, the representative firm fully employs the flow of labour services, evenly distributed among the operations of (infinitesimally small) units which started at time $t-\tau$ (with $0 < \tau < \theta$). The technology is moreover such that techniques are all the more productive as their period of production is higher. This means that the flow of output

[3] In this respect the attention of the reader is drawn to the full section 3 (pages 115-119) which begins with the quotation given above. Before embarking on the mathematical presentation, it is meant to explain how « the relatively definite and very simple concept of the lengthening of the process of production replaces the older, vague, and multiform idea of productivity of capital ». It discusses how an « average length of the period of production » could conceivably be measured in actual economies.

y obtained after a period of length θ by the constant use of a flow of labour $1/\theta$ is an increasing function $f(\theta)$. This function defines the technology.

In order to find the appropriate technique the representative firm, embedded in a stationary competitive market and therefore taking as given the real wage rate *w* and a constant instantaneous real interest rate ρ, has to compute the capital cost of operating any given technique. So, let us do it. Consider first the capital cost to be borne between *t* and *t* + *dt* for financing the labour cost of the units which started operation between $t-\tau$ and $t-\tau + d\tau$. These units employed $Ld\tau/\theta$ unit of labour from date $t-\tau$ to date *t*. Properly capitalized at date *t*, the labour cost to be financed leads to a capital cost equal to *dt* multiplied by :

$$\frac{wL\,d\tau}{\theta}\int_0^\tau \rho\, e^{\rho u}\, du = \frac{wL}{\theta}(e^{\rho\tau}-1)d\tau \qquad (1)$$

Integrating this cost over all units ; i.e. for τ running from 0 to θ, we find the value of the circulating capital *V* to be financed :

$$\rho V = \frac{wL}{\rho\theta}(e^{\rho\theta}-1-\rho\,\theta) \qquad (2)$$

(Note in passing that the aggregate *V* sums up heterogeneous elements : the circulating capitals invested at the various dates $t-\tau$. Problems resulting from such an heterogeneity in the composition of the value of aggregate capital will be more closely examined in section 7).

Collecting terms, we now find that the flow of profit accruing to the representative firm using technique θ is equal to :

$$P = Lf(\theta) - wL - \rho V = L[f(\theta) - \frac{w}{\rho\theta}(e^{\rho\theta}-1)] \qquad (3)$$

The maximising value of θ is then found to be solution of :

$$\rho\theta^2 f'(\theta) = w[1-(1-\rho\theta)e^{\rho\theta}] \qquad (4)$$

The second order inequality holding at a true maximum, namely that the second derivative of *P* with respect to the choice variable θ is non-positive, is easily found to be :

$$f''(\theta) + \frac{2f'(\theta)}{\theta} - \frac{w\rho}{\theta}e^{\rho\theta} \le 0 \qquad (5)$$

To the two equations (2) and (4) we may add, in this competitive economy with constant returns to scale, that the equilibrium value of profit is nil :

$$y = Lf(\theta) = wL + \rho V \qquad (6)$$

Hence, four equations involving five endogenous variables : *y, V,* θ, *w,* ρ and two exogenous elements, the labour supply *L* and the function *f*(.). The missing equation is, of course, that

following from the behaviour of the consumption sector. Wicksell reached his objective, which was to provide a model of the production sector.

Comparative analysis

The purpose, as spelled out in the sentence quoted at the beginning of this section, was to show that, in a family of stationary states of economies having the same production sector, a higher level of capital *V* was associated with a « more round-about method of production », i.e. with a longer production period θ. This follows from the above equations by an argument which is somewhat laborious and will not be reproduced here as such for simplicity.

The argument takes advantage of inequality (5) and shows that, within the one-dimensional family of stationary states, a small increase $d\theta > 0$ of the period of production is associated with a small decrease $d\rho < 0$ of the rate of interest, and with small increases *dy*, *dV* and *dw* of the three other endogenous variables.

In order to show these results, Wicksell used those approximations to equations (2) and (4) which apply, to the first order, when $\rho\theta$ can be taken as small. They are given respectively by :

$$V = \frac{wL\theta}{2} \tag{7}$$

and

$$f'(\theta) = \frac{w\rho}{2} \tag{8}$$

Let us recall the simple argument which can then be made.

Introducing (7) in (6) we derive :

$$f(\theta) = w[1 + \frac{\rho\theta}{2}] \tag{9}$$

Differentiating this equation and taking advantage of (8) we directly find a relation between *dw* and $d\rho$ (i.e. a local expression, in this constant-returns-to-scale technology, of the «factor-price frontier ») :

$$(1 + \frac{\rho\theta}{2})dw + \frac{w\theta}{2}d\rho = 0 \tag{10}$$

Replacing *dw* by the value following from this equation in the equation obtained by differentiation of (8) we reach :

$$\frac{f''}{f'}d\theta = \frac{w}{f}\frac{d\rho}{\rho} \tag{11}$$

Since the second derivative f'' is negative, as soon as the function *f* is concave, a lower real interest rate ρ is indeed associated not only with a higher wage rate *w* but also with a longer period of production θ and a larger capital *V*.

However, we note in passing that it is hard to find in the above argument a convincing proof of the statement that « the role of capital in production is simply *and solely* in making possible the introduction of a longer period of production ». Indeed, the model, in its definition of the function $f(\theta)$, *assumes* that there is no other way in which production could be affected. Less simple models can well open other possibilities. As we shall see in section 6, Wicksell himself explained it. But the model of this section is quite suited to exhibiting why a longer production period will usually require a lower interest rate and a larger amount of capital.

About the choice of a pivotal variable

The presentation just given puts on a par the five endogenous variables appearing in the four-equation model. It so suggests that anyone of these five variables, or any proper combination of them, could serve as a pivot between the model of the production sector worked out by Wicksell and a model of the consumption sector. This is not faithful to the letter of the 1893 book. Indeed the text usually speaks as if the behaviour of consumers would give the value of *V*. The author nowhere suggests that this behaviour in the choice of a value of capital would be insensitive to what are income *y*, the wage rate *w* or the interest rate ρ. But the reader might very well infer that he was not far away from thinking this to be the case.

I frankly believe that it is an unfortunate feature of the text. Indeed, in subsequent writings Wicksell was less systematic. For instance in his solution of the Akerman problem he gave *V, w,* ρ … as functions of θ (*Lectures,* p. 289, his notation differing from the one chosen here).

4. For how long should the input of primary factors mature ?

The main model of production which is exposed in the *Lectures* generalizes, in a sense, the flow input-point output model of *Value*. But Wicksell does not fully solve that more general model. He rather calls in another special case when he wants to more clearly « bring out the importance of the time-element, which is the real kernel of the capital concept ». He then explains : « The simplest conceivable case of the employment of capital… undoubtedly occurs in that form of production where the original factors, land or labour (or both), are *used only once,* as it were in an indivisible moment of time, after which their fruits are spontaneously matured by *free* natural forces ». The examples of laying down of wine for consumption or planting of trees on barren land are given. «In such cases the function of capital is merely to… advance wages or rent for the corresponding period… The length of time will thus be *the only variable dimension of capital.* If, in such a simple case, we are able to deduce the general laws of capital and interest, this deduction may be regarded as an essential ingredient in the explanation of all the more complex phenomena of actual employment of capital » (p. 172).

Actually the mathematical model is simpler than in the previous section and may be developed in the same way. We shall take advantage of the simplicity of the model for

proceeding slowly. We shall so exhibit the nature of the heterogeneity embodied in the value of capital *V*.

At time *t*, if the length of the maturation period is θ, labour has been invested since *t*-θ and has not yet matured in production. The flow of new labour investments keeps going. More precisely in the stationary state we are considering, a labour input equal to $Ld\tau$ took place between the neighbouring times $t-\tau$ and $t-\tau+d\tau$. The original price of this input was *w*. But interest had to be paid on the then invested cost. We may say that, capitalized at time *t*, the price turns out to be :

$$w(\tau) = we^{\rho\tau} \tag{12}$$

Similarly capitalized, the corresponding cost for the same investment period $(t-\tau\,;t-\tau+d\tau)$ is :

$$w(\tau)L\,d\tau$$

Summing the values at time *t* of all investments made since $t-\theta$, we find the aggregate value of the circulating capital :

$$V = L\int_0^{\theta} w(\tau)d\tau = \frac{wL}{\rho}(e^{\rho\theta}-1) \tag{13}$$

Let now $f(\theta)$ be the value of output obtained from an initial unit labour input after a maturation period of length θ, an increasing function of θ. The flow of profit accruing to the representative firm choosing this length is equal to :

$$P = Lf(\theta) - wL - \rho V = L[f(\theta) - we^{\rho\theta}] \tag{14}$$

Thus, the maximizing value of θ is simply given by

$$f'(\theta) = w\,\theta\,e^{\rho\theta} \tag{15}$$

at a point where $f''(\theta)$ is non-positive (let us say negative for simplicity).

After these changes, an argument similar to the one sketched in section 3 can be given, reaching the following comparative analysis property. Of two stationary states in which *L* has the same value and the same function *f*(.) applies, the one with the longer maturation period θ, in comparison to an infinitesimally close value $\theta - d\theta$, has the lower interest rate ρ and the higher output. This argument leads qualitatively to the same result as the one given in *Value*, but from a similarly restrictive assumption : θ is assumed to be « the only variable dimension of capital ».

5. A more general model and the average period of production

But the *Lectures* also introduced a model of production, which may be said more general and later had a life of its own in the reflection of younger economists[4]. This other model should not be neglected in our symposium because it permits us today to realize where stands the limit of validity of the thesis according to which more roundabout methods of production are more productive.

The framework. Its relations to the two foregoing sections

A crucial hypothesis of the new model is the possibility of tracing back to their origin the exact quantities of all primary inputs which entered the production of the final output collected in a current period. For simplicity our notation here will concern the case in which the final output is made of a single good and there is a single primary factor, a homogeneous labour. The case of two primary inputs, labour and land, is used by Wicksell in his presentation of « capitalistic production » (p. 144-166). Like Wicksell there, we shall here take time as being a sequence of periods (…*t*-1, *t*, *t*+1…). A production plan that will mature in period *t* is by hypothesis a quantity of output y_t and a sequence of non-negative quantities $u_{t-\tau,t}$ of the primary input to be used in earlier periods $t-\tau$ ($\tau = 1,...,h$).

The technology applying to the production plan for period *t* is by hypothesis well described by a production function :

$$y_t = F(u_{t-1,t}, u_{t-2,t}, ..., u_{t-h,t}) \qquad (16)$$

Since we are aiming at studying stationary states, we directly wrote the production function *F* as being the same one for all periods *t*. Similarly, the price system to be examined exhibits stationarity in that, the final good being taken as numeraire, this system is fully defined by a constant real wage rate *w* and a constant real interest rate ρ, implying a constant discount factor $\beta = (1+\rho)^{-1}$. Moreover, we assume here, like in the previous two sections, constant returns to scale.

Can we put the model of section 4 into this framework ? Not quite. First, we have to substitute discrete time for continuous time, which was convenient for locating the optimal length of the maturation period, but was probably not so realistic for applications. *The main point is, however, to see whether we can define the production function F* of (16) in such a way that an optimal production plan exists with all its $u_{t-\tau,t}$ equal to zero except for one single τ. This could be achieved, it seems, with the following definition :

$$F = \{\text{there is } \theta \text{ such that } y_t = u_{t-\theta,t}\ f(\theta) = \underset{\theta'=1,2...h}{Max}\ \beta^{\theta-\theta'}\ u_{t-\theta',t}\ \ f(\theta')\} \qquad (17)$$

$f(\theta)$ being precisely the same functions as in section 4. With such a definition, replacing all positive $u_{t-\tau,t}$ by zero except for $u_{t-\theta,t}$, reduces the capitalized cost of production without any reduction in output predicted by (17). We are back to a model whose spirit is close to that of

[4] I shall here particularly refer to J. Hicks (1939) and M. Allais (1947). The latter does not quote Wicksell but acknowledges his debt to G.-H. Bousquet (1936), who has a long chapter on « Time and the equilibrium of production » ; this was directly and indirectly inspired by Wicksell.

section 4. However, in order to lead to the same optimal decision in the choice of θ, we had to somewhat deviate from (16) and to introduce within *F* in the definition (17) the discount factor β, which is a characteristic of the price system, not of the technology.

Moreover, we shall assume later in this section that the production function *F* has well defined first and second order partial derivatives. Clearly, the first derivatives of the function (17) are not everywhere continuous : if at some point there is in *F* just two positive inputs $u_{t-\theta,t}$ and $u_{t-\theta',t}$ which are such that $\beta^{-\theta}\ u_{t-\theta,t} = \beta^{-\theta'}\ u_{t-\theta',t}$, the right-hand derivative of *F* with respect to $u_{t-\theta,t}$ is equal to $f(\theta)$ and the left-hand derivative is equal to zero.

So, strictly speaking, the model of section 4 is not a particular case of the model of this section. The same kind of problems would appear in any attempt to bridge the model of section 3 and the one we are now going to discuss, except that instead of (17) we would have a still more complex definition of the function *F*. However, in a sense, the new specification provides a welcome generalization by allowing for more flexibility and opening more dimensions in the choice of the input time profiles. We should not be surprised when realizing that the specification looked attractive to Wicksell, as well as to others in the interwar period and later.

Solution of the model

Formal solution of the model applying under perfect competition is easy to derive as soon as the function *F* is assumed to be twice differentiable. When deciding its production plan in period *t – h,* the price-taking representative firm chooses the input series $u_{t-\tau,t}$ so as to maximize the present value :

$$\beta^h\ F(u_{t-1,t},...,u_{t-h,t}) - w\sum_{\tau=1}^{h}\beta^{h-\tau}\ u_{t-\tau,t} \tag{18}$$

The maximizing inputs have to be such that, for all τ :

$$\frac{\partial F}{\partial u_{t-\tau,t}} = w\beta^{-\tau} \tag{19}$$

and that the matrix *H* of the second derivatives of *F*, often called the Hessian of *F*, be negative semi-definite. Hence, the marginal rate of substitution between $u_{t-\tau,t}$ and $u_{t-\tau+1,t}$ is equal[5] to $1+\rho$.

The *h* equations (19) do not provide a full determination of the *h* inputs $u_{t-\tau,t}$ since, as is well known, they are not linearly independent with a constant-returns-to-scale production function *F*. Indeed, we need to also account for the equilibrium of the labour market where supply is the exogenous quantity *L* :

$$\sum_{\tau=1}^{h} u_{t-\tau,t} = L \tag{20}$$

[5] See *Lectures,* p. 156.

Normally the system (19)-(20) gives a full determination of the input vector. The solution is clearly independent of *t*. Here it will be convenient to define v^* as the vector of which the *h* components are given by the solution $(u_{t-1,t},...,u_{t-h,t})$ of (19)-(20).

For comparative analysis we shall have to consider the effect dv^* on v^* of infinitesimal changes *dw* in *w* and $d\beta$ in β. This will be given by differentiation of system (19)-(20), in which will appear the Hessian *H*, which normally has rank *h*-1. Differentiation of (19)-(20) leads to :

$$H\ dv^* = \lambda\ dw - w\ \mu \frac{d\beta}{\beta} \tag{21}$$

$$\sum_{\tau-1}^{h} dv_\tau^* = 0 \tag{22}$$

where λ and μ are the vectors with components :

$$\lambda_\tau = \beta^{-\tau} \qquad \mu_\tau = \tau\beta^{-\tau} \tag{23}$$

As long as *H* has rank *h*-1, the system (21)-(22) has a unique solution.

Comparative analysis and the average period of production

Can we obtain from this model a generalization of the property according to which, in the family of stationary states compatible with the given technology, a lower real interest rate is associated with the choice of more roundabout methods of production ? John Hicks presented in *Value and Capital* (1939) the outcome of clever intuitions leading to such a generalization : a lower interest rate is indeed associated with a longer average period of production, but under two conditions, the first concerning the definition of the average period of production, the second the meaning of « a longer average period ».

Quite naturally the average period of production is a weighted average of the numbers, *h*, *h*-1,…, 1 characterizing the length of time τ between the input $u_{t-\tau,t}$ and the output y_t. The weights should reflect in particular the relative importances of the various quantities $v_\tau^* = u_{t-\tau,t}$. But Hicks' intuition suggested that these weight should not be exactly proportional to these quantities. The weighted average had rather to be :

$$\bar{\theta} = \sum_{\tau=1}^{h} \tau\ \beta^{-\tau}\ v_\tau^* \Big/ \sum_{\tau=1}^{h} \beta^{-\tau}\ v_\tau^* \tag{24}$$

For instance in the flow input-point output model of section 3, the average period of production so defined with the continuous representation of time was not equal to $\theta/2$, as would be spontaneously said, but rather to :

$$\bar{\theta} = \frac{1-(1-\rho\theta)e^{\rho\theta}}{\rho[e^{\rho\theta}-1]} \tag{25}$$

which is approximately equal to

$$\frac{\theta}{2}\left[1+\rho\,\frac{\theta}{6}\right] \tag{26}$$

Early inputs weigh more in the calculation than late inputs.

Hicks' intuition also suggested, however, that the change in $\bar{\theta}$ given by (24) was *not* the correct indicator to use in comparative assessments about the lengthening of the average period of production under varying values of ρ, or equivalently β. Indeed, the change in (24) would reflect not only the change in the time-profile of the inputs, which is what lengthening means, but also the change in β. The correct indicator of lengthening must therefore be computed with unchanging β.

In this paper the distinction is important because it is related to one of the fundamental concerns expressed by Wicksell, a concern which may be thought to still remain challenging in the pure theory of capital (see section 7 here). So, let us quote Hicks at this point : « If the average period changes, without the rate of interest having changed, it must indicate a change in the stream [of inputs] ; but if it changes, when the rate of interest changes, this need not indicate any change in the stream at all. Consequently, even when we are considering the effect of changes in the rate of interest on the production plan, we must not allow the rate of interest which we use in the *calculation* of the average period to be changed » (p. 220 in Hicks, 1939).

This methodological point being understood, it is fairly easy to prove the relevant comparative analysis property. This concerns the infinitesimal change in the equilibrium production plan after an infinitesimal change $d\beta$ in the real discount factor β. We may deduce from (24) the following formula defining the average period of production $\bar{\theta}$ before the change in β :

$$\sum_{\tau=1}^{h}(\bar{\theta}-\tau)\beta^{-\tau}\ v_\tau^* = 0 \tag{27}$$

Similarly if $\bar{\theta}+\hat{d}\,\bar{\theta}$ denotes the average period of production after the change in β but computed with the discount factors used before the change, we may write :

$$\sum_{\tau=1}^{h}(\bar{\theta}+\hat{d}\,\bar{\theta}-\tau)\beta^{-\tau}(v_\tau^*+dv_\tau^*) = 0 \tag{28}$$

in which the infinitesimal changes dv_τ^* are given by equations (21)-(22). Substracting (27) from (28) and neglecting the products $\hat{d}\,\bar{\theta}.dv_\tau^*$ which are second-order small, we find :

$$\hat{d}\,\bar{\theta}\sum_{\tau=1}^{h}\beta^{-\tau}\ v_\tau^* + \sum_{\tau=1}^{h}(\bar{\theta}-\tau)\beta^{-\tau}dv_\tau^* = 0 \tag{29}$$

In order to draw a conclusion from this equation let us first note that constant returns to scale and equation (19) imply :

$$y = w \sum_{\tau=1}^{h} \beta^{-\tau} \; v_{\tau}^{*} \tag{30}$$

In turn, if we differentiate both sides of this equation, the left-hand side as *dF* and the right-hand side as written above we find :

$$0 = dw \sum_{\tau=1}^{h} \beta^{-\tau} \; v_{\tau}^{*} - \frac{w d\beta}{\beta} \sum_{\tau=1}^{h} \tau \beta^{-\tau} \; v_{\tau}^{*} \tag{31}$$

Given the definition of $\bar{\theta}$ by (27), this is equivalent to :

$$\frac{dw}{w} = \bar{\theta} \; \frac{d\beta}{\beta} \tag{32}$$

which is nothing else than the local expression of the « factor price frontier ».

Equation (21) then writes as :

$$\frac{1}{w} H \; dv^{*} = [\bar{\theta}\lambda - \mu] \frac{d\beta}{\beta} \tag{33}$$

As long as the negative semi-definite matrix *H* has rank *h*-1, it has just one characteristic vector *z* such that *Hz* = 0. All the components of this vector are non-negative (they are proportional to those of v^{*}). Because of (22), dv^{*} cannot be colinear to *z*. This implies that the quadratic form $dv^{*'} \; H \; dv^{*}$ is negative, which together with the last equation implies :

$$\frac{d\beta}{\beta} . \sum_{\tau=1}^{h} (\bar{\theta} - \tau) \beta^{-\tau} \; dv_{\tau}^{*} < 0 \tag{34}$$

Multiplying equation (29) by $\frac{d\beta}{\beta}$ and taking account of (30) and (34), we directly find :

$$\frac{y}{w} \hat{d} \; \bar{\theta} . \frac{d\beta}{\beta} > 0 \tag{35}$$

A decrease in the real interest rate ρ, i.e. an increase in β, is associated with a lengthening of the average period of production, given what we mean by such a lengthening.

This is a significant generalization of the results recalled in the previous two sections because, besides scale of operation, the technology covered by (16) now allows for $h - 1$ dimensions of variation in inputs. Restricting to stationary competitive equilibria, reduces this dimensionality to just one, since the two parameters of the price system are bound by the «factor price frontier » (32). It is interesting to know that the average period of production, a

measure of the degree of roundaboutness, contravaries with the interest rate along this dimension.

Readers of Hicks (1939) as well as Hicks himself may, however, have been misled into giving to the property more generality than it really has. The restriction to a single produced good is essential and this is not enough stressed in *Value and Capital*. Indeed, the discussion of the production plan is split between two chapters : chapter XVI discusses the effect of changes in relative prices, chapter XVII turns to the theory of interest changes[6] but ignores possible induced changes in relative prices, which is a disturbing restriction for a theory aiming at characterizing general equilibria. Working on the Akerman problem Wicksell had, however, already faced the difficulty, as I am now going to recall.

In the last section of this article I shall moreover still comment on how the approach used by Wicksell and his followers compares with an alternative approach which is now much more commonly used in capital theory.

6. A two-sector model with fixed capital

Wicksell had insisted in *Value*, and in the *Lectures* as originally published, that his analysis of capitalistic production concerned only «circulating capital » not fixed capital. Beginning in 1923 his review of Gustaf Akerman's *Realkapital und Kapitalzins,* a review published at the end of the English edition of the *Lectures,* Wicksell writes : the object of the book is to investigate the *co-operation* of social *durable* capital with free uninvested labour in production. This problem is clearly of great practical significance - no doubt much more so than the problems dealt with by Jevons and Böhm-Bawerk. They concentrated on the capitalistic process of production, in which labour resources (and probably land resources) ripened into immediate consumption goods, or what the author calls «variable capital ». But his problem is so complex that the vast majority of economists, including the reviewer, have almost entirely passed it by as being much too difficult to be susceptible to analysis ». The review covers 42 pages, of which 26 are devoted to «A mathematical analysis of Dr. Akerman's problem » in which Wicksell presents « a mathematical solution of the problem ». The solution is found in the framework of a model, which we are now going to consider using a notation consistent with that of the previous pages and with current practice.

There are two produced commodities, a capital-good, initially called « an axe » by Wicksell (we shall rather speak of a machine), and a consumption-good, serving as numeraire. The machine is produced directly and instantaneously from labour. Depending on the labour input z used in its production, a machine will last a more or less long period θ, providing all along a constant flow of services, valued at the real price v for a unit period in a stationary state.

Wicksell gives to the constant-returns-to-scale production function of the capital-good sector, sector 1 say, the following particular form :

$$z = k\,\theta^{\nu} \tag{36}$$

where k is a positive constant and ν a positive parameter smaller than one.

[6] Hicks even writes : «I believe I have discovered such a theory » (p. 213).

Equality between the cost of production of the machine and the discounted value of the flow of its services (which applies under perfect competition) leads to :

$$wz = \frac{v}{\rho}[1 - e^{-\rho\theta}] \tag{37}$$

Taking (36) into account we see that perfect competition and price-taking behaviour in sector 1 implies that θ is chosen so as to maximize :

$$\frac{v}{\rho}[1 - e^{-\rho\theta}] - wk\theta^{v}$$

which leads to :

$$v\, e^{-\rho\theta} = v\; w\, k\; \theta^{v-1} = \frac{v}{\theta} w\, z$$

Taking (37) into account, we find the simple equation :

$$\frac{\rho\theta}{v} = e^{\rho\theta} - 1 \tag{38}$$

which shows that the product $\rho\theta$ depends only on the value of the parameter v .

The production of the flow of the consumption-good in sector 2 is also instantaneous. It uses labour and the service provided by machines. Wicksell gives to the production function of sector 2 the form of a constant-returns-to-scale Cobb-Douglas function, namely in a stationary state :

$$y_2 = c\; L_2^{\alpha}\; J^{1-\alpha} \tag{39}$$

where y_2, L_2 and J are respectively the output flow, the input of labour services and the input of the service provided by the standing machines, which may be identified with the number of these machines ; c is a positive constant and α a positive parameter smaller than one. Under perfect competition the production plan of sector 2 is bound not only by (39) but also by the now familiar equalities between unit costs and marginal productivities :

$$w = \frac{\alpha\, y_2}{L_2} \qquad\qquad v = \frac{(1-\alpha) y_2}{J} \tag{40}$$

The constraints on stationary states implies also consistency between the exogenous labour supply L and the sum of the flows of labour inputs :

$$L = \frac{zJ}{\theta} + L_2 \tag{41}$$

(The flow of the new machines to be produced for replacement of those worn-out is equal to J/θ).

Moreover the total current value of all vintages of machines of the various ages is easily computed to be :

$$V = v\ J\ \frac{e^{-\rho\theta} - 1 + \rho\theta}{\rho^2\theta} \tag{42}$$

The system of the eight equations (36) to (42) contains nine endogenous variables z, $\theta, v, w, \rho, y_2, L_2, J, V$. The family of stationary states, which are compatible with this two-sector model of production and with given values of L, ν, α, k and c, has one degree of freedom, which depends on the behaviour of the consumption sector and may be taken here as being the real interest rate ρ. Wicksell solves the model and finds that, in the family in question, low interest rate ρ is associated not only with a long period of service θ of the machines, with a large number J of machines, with a high wage rate w and a high production of the consumption-good y_2, but also with a low price v of the services of the machines and nevertheless a high value V of capital[7].

However, and this is very revealing of his reactions when faced with a difficult problem, Wicksell is not fully satisfied with the results. Indeed, he notes that the ratio L_2 / L, characterizing the allocation of labour between the consumption-good sector (as « free uninvested labour ») and the capital-good sector, has the same value in all stationary states. This astonishing result, he diagnoses, reflects « the assumptions we made (1) for the technical conditions under which our capital-goods are manufactured, and (2) for their co-operation with free labour in the production of consumption goods » (p. 288). Thus, Wicksell stands as a remote ancestor of those modern economists who, after having been attracted by « the log-linear economy » because of its convenience for general equilibrium computations, also discovered that it was very special and even often misleading in some of its implications.

This stimulated the reflections in which Wicksell distinguished the « breadth » and « height » dimensions of capital. He had spent his major efforts in the pure theory of capital for showing the importance of the height dimension either for circulating capital with the period of production or for fixed capital with the durability of machines. He was finding that the allocation of labour between the two sectors did not change with the level of the interest rate, which he interpreted as meaning that the breadth of capital did not change[8]. But he was less and less inclined to neglect possible increases in the breadth of capital (we more commonly speak today of «capital widening » as against « capital deepening »). This was partly due to his concern about business cycles, in which he analysed the first impact of an increase in savings to be an increase in the breadth of capital, the increase in the height being more progressive.

[7] We note here in passing that the simultaneous variations in v and w from an assumed variation in ρ exhibit a case in which variations in relative prices cannot be neglected for the study of the effects of variations in the interest rate. The separability assumed by Hicks does not hold.

[8] Note, however, that a lower interest meant a larger number of machines in service.

But, even for stationary states, he writes the following two sentences at the end of the *Lectures*, within a literary argument showing that an increase in capital must, as a general rule independently of his particular model, always be associated with an increase in the volume of production : «an increase in capital may very well occur with an accompanying *fall* in the breadth dimension… Let us now take the commonest instance in which machine - capital increases in breadth as well as in height… » (p. 297-8). Thus, according to Wicksell's intuition, two possibilities exist for deviations from the comparative long-run properties of the log-linear economy, either a decrease or an increase in the breath dimension. But the latter is the most common.

7. Two of Wicksell's legacies

After this survey of Wicksell's contributions to the pure theory of capital, complemented by the treatment of the average period of production in Hicks' *Value and Capital,* we are led to reflect about their impact on the subsequent development of the theory. Comments here will relate to just two features of this development, the first concerning the choice of the production model, the second the study of the so-called Wicksell effects.

About the choice of the production model

In Chapter 1 of *Capital and Time* (1973) Hicks brings into contrast two methods for representing production in capital theory : the «method of von Neumann » (production lasts just one elementary period, inputs occurring at the beginning, outputs at the end) and the «method of separable elementary processes » (a process uses only primary inputs and produces only outputs directly used in consumption)[9]. According to this dichotomy, it is fair to say that the method of von Neumann, which in principle distinguishes as many intermediate goods and their markets as required for appropriate modelling, entirely dominates modern research practice. Should the method of separable elementary processes, which was used by Wicksell and his immediate followers, be now definitely discarded ?

Clearly Hicks thought that would be premature. He did not ignore, however, the difficulty with that method in applications where primary inputs must be identified as being allocated to specific final outputs[10]. But he thought the method to be suitable for correctly drawing the consequences of «extensive *complementarities over time* » (his italics). Indeed, we must agree that, as now practised with in particular strong assumptions of time separability, von Neumann method makes « the economy too flexible ». This may not much matter when attention is limited to stationary states. But Hicks was interested also in what he called « the traverse » and had been called «capital accumulation » by Wicksell. Hicks was similarly interested in tracing the response of the economy to a technical innovation from its introduction to its generalized use. And so had been Wicksell, in parts of his writings, which did not concern the stationary states discussed in this paper. Thus, it is not extravagant to imagine that some real progress will someday emerge again from application of the method of separable elementary processes.

With this possible prospect in mind we ought to focus some of our attention on the true nature and extend of complementarities and substitutions over time. The models discussed in this paper provide just a sample of what could be discussed. Substitutions are

[9] For a fuller discussion, see my Hicks Lecture, Malinvaud (1986).
[10] Neither did Wicksell ignore the difficulty, as can be seen by the long paragraph p. 165-6 of the *Lectures.*

stressed in section 5 as soon as twice differentiability is assumed. In contrast, full complementarity is assumed within the flow of labour services used by a given technique in section 3 or within the flow of services provided by a machine in section 6. How are our visions about the role of physical capital affected when we move from one specification to another ?

I hasten to add that a return to models similar to those used by Wicksell would be fully compatible with the abstract model of the intertemporal general competitive equilibrium, which now provides the foundations of the pure theory of capital. The basic principle of this abstract model is to distinguish as many commodities as needed to make two units of the same commodity perfectly interchangeable. For instance with the models of sections 3 and 4 we have to identify commodities corresponding to previously invested labour, each one with its price given by (12) and this for all positive τ .

With the model of section 6 dealing with the Akerman problem, identification of commodities has to distinguish as many types of machines as they differ in the length θ of their period of utilization and in their age $a = \theta - \tau$. In a stationary equilibrium the price $q(\theta, a)$ of a machine of age *a* and period θ is obtained as the discounted value of its future services. With the notation of section 6, this leads to[11] :

$$q(\theta, a) = \frac{v}{\rho}[1 - e^{-\rho(\theta - a)}] \tag{43}$$

this being defined for all positive θ and all *a* in the interval $[0, \theta]$. And indeed the value *V* of the stock of machines existing at any time, as given by equation (42), is equal to :

$$V = \frac{J}{\theta} \int_0^\theta q(\theta, a)\, da \tag{44}$$

since there are Jda/θ machines of age *a* in the interval [*a*, *a* + *da*].

It appears here that fundamentally the model of section 6 contains many capital goods, each one with its price $q(\theta, a)$. Application of the now common «method of von Neumaun » to the search for properties of comparative analysis in capital theory has shown that precisely the multiplicity of capital goods was the source of many complications[12]. Perhaps, the « method of separable elementary processes » will more easily uncover some interesting new properties.

Wicksell effects, the marginal productivity of capital and aggregation.

The second group of reflections to be offered at the end of this article may start with a reference that was made to a « Wicksell effect » at the time of the Cambridge controversies. I do not recommend the use of this phrase because I found it too often used in contexts where

[11] The formula is simple in a stationary equilibrium, since it involves only two other price variables *v* and ρ . It would be much more complex in a non-stationary equilibrium.

[12] For a flavour of these complications and bibliographical references, it may be proper here to mention E. Burmeister (1987).

its meaning remained obscure to me. I also saw that different authors gave it different definitions. I may follow here Samuelson (1987) who defines the Wicksell effect as :

$$\rho - \frac{d\left[\sum_{i=1}^{n} p_i y_i\right]}{d\left[\sum_{i=1}^{n} p_i K_i\right]} \tag{45}$$

referring implicitly to a model in which there would be n commodities (i = 1, 2…n) with respective prices p_i, the quantities of output and capital of these commodities being y_i and K_i. Moreover d is meant to be the operator defining infinitesimal changes when some such change is exogenously introduced in the model. Note that this so-called effect has no precise meaning unless the model and the exogenous change are defined.

For instance with the model of section 3 we may write (45) as :

$$\rho - \frac{dy}{dV} \tag{46}$$

and consider the value of the difference for an infinitesimal variation $\mathrm{d}\theta$ taken as exogenous, the parameters and the exogenous L remaining unchanged. (Since there is just one degree of freedom in the family of stationary states, the exogenous change could concern any other endogenous variable ; it could as well be dV). The equations displayed in section 3 actually lead to :

$$\frac{dy}{dV} = \rho\left[1 - \frac{\theta^2 f''}{w}\right]^{-1} \tag{47}$$

This corresponds exactly to the equation derived by Wicksell in 1893 (see *Value,* bottom of page 138). Since the second derivative f'' is negative the ratio dy/dV is smaller than ρ. The « Wicksell effect » is positive.

Why did such a result attract attention in capital theory ? Without claiming to give a complete answer to this question, I shall speculate about what such an answer might be.

I must first clear the ground and flatly declare that many economists erroneously found in Wicksell's results a contradiction with neoclassical economics, which was said to assert as a basic tenet that the net marginal productivity of capital was equal to the real interest rate. I need not enter here into a detailed examination of what « neoclassical economics » should exactly say, or of the errors to be found in the publications of some economists ranked as neoclassical. It is clear enough that the left-hand side of equation (47) is not a relevant measure of the marginal productivity of capital in this context : its denominator is a change in the value of capital not in the volume of capital, as would be required in order to make sense of the equality between the real interest rate and the marginal productivity in question.

In our reflections here it may be interesting to figure out what Wicksell would have found in his analysis of the Akerman problem if he had tried to derive the equation applying to what he called the physical marginal productivity of capital. Reasoning as Hicks did about the model explained here in section 5, Wicksell would have argued (1) that the volume *K* of capital must be defined by aggregation of the numbers J/θ of the machines of the various ages $a = \theta - \tau$ using proper weights, namely the respective prices $q(\theta, a)$, (2) that for comparative assessments in the neighbourhood of a given stationary state where by definition *K = V*, the physical marginal productivity of capital should not be defined with respect to infinitesimal changes in *V*, from one stationary state to another neighbouring stationary state, but to changes $\hat{d}K$ in *K* as computed with unchanged weights.

Let us apply this principle. At the level of the elementary commodities the physical change is from J/θ machines of each age, all made in order to serve during a period of length θ, to $J/\theta + d(J/\theta)$ machines of each age, all made in order to serve during a period of length $\theta + d\theta$. Hence :

$$\hat{d}K = \left[\frac{J}{\theta} + d(J/\theta)\right]\int_0^{\theta+d\theta} \hat{q}(\theta + d\theta, a)\, da - \frac{J}{\theta}\int_0^{\theta} q(\theta, a)\, da \qquad (48)$$

where the $\hat{q}(\theta + d\theta, a)$ are the prices which would have been allocated to machines made in order to serve during a period of length $\theta + d\theta$ if they had existed in the initial stationary state. For any age *a* in the interval [0, θ] the price in question would have been :

$$\hat{q}(\theta + d\theta, a) = q(\theta, a) + ve^{-\rho(\theta - a)}\, d\theta \qquad (49)$$

(with respect to (43) *v* and ρ are kept unchanged and θ is changed to $\theta + \mathrm{d}\theta$). Consider the integral involving this price in (48) and note (1) that the part concerning the interval $[\theta, \theta + d\theta]$ is second order small in terms of $d\theta$, hence it is negligible, (2) that moreover :

$$\int_0^{\theta} q(\theta, a)\, da = \frac{v}{\rho^2}[e^{-\rho\theta} - 1 + \rho\theta] \qquad \int_0^{\theta} e^{-\rho(\theta - a)}\, da = \frac{1}{\rho}[1 - e^{-\rho\theta}] \qquad (50)$$

Hence (48) may be written as :

$$\hat{d}K = \frac{vJ}{\rho\theta}[1 - e^{-\rho\theta}]d\theta + d(J/\theta)\frac{v}{\rho^2}[e^{-\rho\theta} - 1 + \rho\theta] \qquad (51)$$

The solution worked out by Wicksell (see *Lectures* bottom of page 289) implies :

$$d(J/\theta) = -v\frac{J}{\theta}.\frac{d\theta}{\theta} \qquad (52)$$

where v is the parameter of the production function (36) applying to the sector producing the machines. It then follows that (51) leads to :

$$\hat{d}K = \frac{vJ}{\rho}\left\{1-\nu-e^{-\rho\theta}\left[1-\frac{\nu}{\rho\theta}(e^{\rho\theta}-1)\right]\right\}\frac{d\theta}{\theta} \qquad (53)$$

But equation (38) shows that the square bracket is equal to zero. Hence :

$$\hat{d}K = (1-\nu)\frac{vJ}{\rho}.\frac{d\theta}{\theta} \qquad (54)$$

The solution worked out by Wicksell also implies :

$$dy_2 = (1-\nu)(1-\alpha)y_2\frac{d\theta}{\theta} \qquad (55)$$

Hence the physical net marginal productivity of capital is equal to :

$$\frac{dy_2}{\hat{d}K} = \frac{\rho(1-\alpha)y_2}{vJ} \qquad (56)$$

which, in a stationary equilibrium, is equal precisely to the real rate of interest because of (40).

Wicksell did not develop this argument. But I am sure he would have been interested in it, and more generally in the possibility of making sense, in his models, of the concept of «increase in the volume of social capital », which he actually avoided. Indeed, he was well aware of the importance of the distinction between physical and value productivity when he wrote the following sentences in a footnote of the *Lectures* (p. 259) : «Böhm-Bawerk's criticism is in effect identical with his celebrated objection against all « productivity theorists », who in his opinion constantly confuse physical and value productivity… At the very most this confusion is nothing more than a methodological error. In the first approach to the solution of the problem of production and distribution, it is permissible, if not advisable, to consider the prices of commodities as constant… ; in the same way, we regard production as constant in the first stage of the solution of the problem of pricing. It is only at a later stage that we should combine both these approximations in order to obtain the final solution of the problem ».

Probably if Wicksell avoided to speak of increases in the volume of social capital in his approach relying on separable elementary processes, it is because he realized that definition of the concept was difficult in that approach. We indeed read in an incidental passage of the *Lectures* (p. 165) a reference to « an increase in the volume of social capital » and, a little later : « It may be difficult - if not impossible – to define this concept of social capital with absolute precision, as a definite quantity. In reality, it is rather a *complex* of quantities ». Indeed, the abstract theory of the intertemporal general equilibrium was not as developed in Wicksell times as it now is.

In 1893 when presenting his result, written here as equation (47), and showing that the marginal « value productivity » of capital was lower than the interest rate, Wicksell had a particular motivation, which a strong group of British economists working at Cambridge in

the nineteen fifties found consonant with their concerns. This should not be forgotten because it explains the success to the phrase « Wicksell effect ».

My memory of the writings on capital theory by the English Cambridge school is incomplete. Reading it I was often stopped by assertions that I thought to be misleading. Some came from the confusion of physical productivity with value productivity, to which I just referred. Others arose from a misreading of the then modern welfare economics. As Samuelson (1987) wrote : « Recognizing Wicksell effects… is not to agree with the frequently met notion that, in consequence, the steady-state interest rate of perfect competition can lack intertemporal Pareto-optimality. Actually, no matter what… Wicksell effects are present, the competitive equilibrium does support *intertemporal production-and-consumption efficiency* » (he might have added « when external effects are absent »).

Similarly was misleading the then frequently read notion that Wicksell effects were a product of aggregation. For instance Carl Uhr (1951), when inventing the phrase « Wicksell effect », wrote that the equality between the marginal productivity of capital and the rate of interest applies « only at the private or micro-economic level » (p. 851). No, in models like the ones used by Wicksell, it applies to the physical marginal productivity at both the macroeconomic and microeconomic levels (remember that in perfect competition microeconomic agents are price-takers). This is not to deny the possible importance of aggregation effects, usually arising from microeconomic heterogeneity. But this aspect was not studied by Wicksell.

About his result on value marginal productivity and its importance Wicksell actually wrote : « One could… be led… to believe that… the surplus return obtained through [an increase in the national capital], divided by the capital increase in question, will give us approximately the level of interest. This would be decidedly wrong. The result of this division sum is… always *smaller* than the interest… This is connected with the fact that [the] increase in the national capital is accompanied by an increase in wages which partially swallows it up » (top of page 137). Uhr (1951) rightly draws attention to the importance of this result in the context of the theory of production and distribution, as it developed during the last decades of the nineteenth century and the first half of the twentieth. The concern was to know whether a continuous net accumulation of capital would generate a « tendency toward a zero interest rate and, presumably, a stationary society » (bottom of page 850). But, as Uhr notes, Wicksell had not been the first economist to make the point. At least in the last edition (1871) of his celebrated book whose first edition appeared in 1848, John-Stuart Mill had written among his fundamental propositions respecting capital : « Increase of capital gives increased employment to labour, without assignable bounds » (title of section 3 of chapter V in the table of contents). In the argument the increase in wages induced by saving played the main part. I may add a somewhat mischievous comment and recall that Robert Solow (1956) begins his presentation of neoclassical growth by pointing to the inadequacy of the crucial Harrod-Domar strict complementarity hypothesis, arguing that substitutability between productive factors will lead in particular to a progressive increase in wages. There Solow was a Wicksellian.

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