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Perturbation Approach Applied to the Asymptotic Study of Random Operators

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Perturbation approach applied to the asymptotic study of random operators.

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Résumé : Nous montrons que, pour les principaux types de theorèmes limites (loi des grands nombres, théorème central limite, principe de grandes déviations et loi du logarithme itéré), des résultats asymptotiques pour des opérateurs aléatoires auto-adjoints conduisent à des résultats du même type concernant leurs valeurs propres et les projecteurs qui y sont associés. Quelques applications statistiques sont mentionnées.

Abstract : We prove that, for the main kind of limit theorems (law of large numbers, central limit theorem, large deviation principle, law of the iterated logarithm) asymptotic results for self-adjoint random operators yield equivalent results for their eigenvalues and associated projectors. Statistical applications are mentioned.

Mots clés : Opérateurs aléatoires, Théorèmes limites, Principes de transferts, Analyse en composantes principales fonctionnelles.

Keywords : Random operators, Limit theorems, Transfert principle, Functional principal components analysis.

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1 Introduction and statement of the mains results

Let H be a separable Hilbert Space (with norm $\|\cdot\|$ and scalar product $< .,. >$). Denote by $\mathcal{L}(H)$ the separable Banach space of bounded linear operators from H to H endowed with the norm

$$\|\cdot\|_{\mathcal{L}} : x \in \mathcal{L}(H) \mapsto \sup_{\|h\|\leq 1} \|x(h)\|,$$

and define the subspace of $\mathcal{L}$ of Hilbert-Schmidt operators,

$$\mathcal{S} = \left\{ s \in \mathcal{L}(H) : \sum_{p\in\mathbb{N}} \|s(e_p)\|^2 < \infty \right\},$$

where $(e_p)_{p\in\mathbb{N}}$ is any complete orthonormal system in H. It is well known (see [5]) that if we define the scalar product

$$\langle s,t\rangle_{\mathcal{S}} = \sum_{p\in\mathbb{N}} \langle s(e_p), t(e_p)\rangle, \tag{1}$$

$\mathcal{S}$ becomes a separable Hilbert space.
Let C be a self-adjoint bounded operator and consider a sequence (C_n) of random self-adjoint elements of $\mathcal{L}(H)$ defined on a common probability space $(\Omega, \mathcal{A}, \mathbb{P})$.
Since C (resp. C_n) is bounded and self-adjoint, its eigen-values $(\lambda_k)_{k\geq 1}$ (resp. $(\lambda_{k,n})_{k\geq 1}$) are uniformly bounded real numbers. Without loss of generality, we assume that $(\lambda_k)_{k\geq 1}$ and $(\lambda_{k,n})_{k\geq 1}$ are non-increasing sequences. For every $k \geq 1$, we denote by m_k (resp. $m_{k,n}$) the multiplicity degree of λ_k (resp. $\lambda_{k,n}$) and by Π_k (resp. $\Pi_{k,n}$) the associated projector of λ_k (resp. $\lambda_{k,n}$).
In the following, $(C_n)_{n\geq 1}$ will be considered as a sequence of estimators of C and our aim is to study how several limit theorems caracterizing the convergence of C_n to C can be used to infer informations about the convergence of $(\lambda_{k,n})_{n\geq 1}$ to λ_k and of $(\Pi_{k,n})_{n\geq 1}$ to Π_k.
Many papers deal with this topic since applications are possible in the area of principal component or canonical analysis for random vectors or functions. In particular, when C is the covariance operator of a process, the estimation of the empirical eigen-elements of C is of great interest since it is connected with data analysis of the observed process (see [12]). The sequence C_n is consequently often the empirical covariance operator of a sample for which several dependence assumptions have been considered. For instance, in their pioneering work [3], the authors studied almost sure convergence and central limit theorem in the case where C_n is the empirical covariance operator of a sample of i.i.d. random functions (C_n then becomes a sum of i.i.d. random operators). In [2], the hilbertian autoregressive context is investigated by martingale techniques, still for almost sure convergence and central limit theorem. These results are generalised in [9] to linear hilbertian processes. In a second time, all these authors

obtained almost sure and weak convergence theorems for the eigen-elements of C. In [10] and in [11] moderate deviations principles and compact laws of the iterated logarithm for hilbertian autoregressive processes are considered. The main goal of this paper is to provide some general results focussing on the transfert procedure beetween limit theorems caracterising the convergence of C_n to C and the same kind of limit theorems for their eigen-elements. We refer to [13] for some interesting results in the finite dimensional case (matrices instead of operators) in the context of the central limit theorem. Note anyway that the methods of the proof do not rely on an improved version of the "delta-method" since we do not use Taylor expansions. The only background needed is very basic facts in perturbation theory (see [8] or the first chapter of [6]).
The different kind of limit theorems considered are listed below.

Definition 1 *Let $(E, \|\cdot\|_E)$ be a Banach space with Borel σ-algebra $\mathcal{B}(E)$ and consider a sequence of E-valued random variable $(W_n)_{n\geq 1}$ defined on a common probability space $(\Omega, \mathcal{A}, \mathbb{P})$.*
i) $(W_n)_{n\geq 1}$ converges almost surely to 0 in $(E, \|\cdot\|_E)$ whenever

$$\|W_n\|_E \to 0 \text{ a.s.}$$

ii) $(W_n)_{n\geq 1}$ converges in law in $(E, \|\cdot\|_E)$ to the limit law G whenever for every $A \in \mathcal{B}(E)$,

$$\limsup_{n\to\infty} \mathbb{P}(W_n \in A) \leq \mathrm{G}\left(\overline{A}\right),$$

were G is a probability measure on $\mathcal{B}(E)$ (here and after, (resp. $\mathring{A}$) denotes the closure (resp. interior) of A in E).
iii) $(W_n)_{n\geq 1}$ follows the large deviation principle in $(E, \|\cdot\|_E)$ with rate function J and speed $v_n \downarrow 0$ whenever for every $A \in \mathcal{B}(E)$,

$$\begin{aligned} -\inf\left\{\mathrm{J}(x) : x \in \mathring{A}\right\} &\leq \liminf_{n\to\infty} v_n \log \mathbb{P}(W_n \in A) \\ &\leq \limsup_{n\to\infty} v_n \log \mathbb{P}(W_n \in A) \leq -\inf\left\{\mathrm{J}(x) : x \in \overline{A}\right\}, \end{aligned}$$

where $\mathrm{J} : E \to [0, +\infty]$ is such that for all $\alpha > 0$, $\{\mathrm{J} \leq \alpha\}$ is compact in E.
iv) $(W_n)_{n\geq 1}$ is almost surely relatively compact in $(E, \|\cdot\|_E)$ with limit set $K \subset E$ whenever
a) K is compact
b)

$$\limsup_{n\to\infty} \inf_{x\in K} \|W_n - x\|_E = 0 \text{ a.s.}$$

c) For all $x \in K$,

$$\liminf_{n\to\infty} \|W_n - x\|_E = 0 \text{ a.s.}$$

The next lemma contains some well known facts :

Lemma 2 *Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be two Banach spaces and let $\varphi : E \to F$ be a continuous function. The following hold :*
i) If $(W_n)_{n\geq 1}$ converges almost surely to 0 in $(E, \|\cdot\|_E)$ then $(\varphi(W_n))_{n\geq 1}$ converges almost surely to 0 in $(F, \|\cdot\|_F)$
ii) If $(W_n)_{n\geq 1}$ converges in law in $(E, \|\cdot\|_E)$ to the limit law G, then $(\varphi(W_n))_{n\geq 1}$ converges in law in $(F, \|\cdot\|_F)$ to the limit law $\mathrm{G} \circ \varphi^{-1}$.
iii) If $(W_n)_{n\geq 1}$ follows the large deviation principle in $(E, \|\cdot\|_E)$ with speed $v_n \downarrow 0$ and rate function J, then $(\varphi(W_n))_{n\geq 1}$ follows the large deviation principle in $(F, \|\cdot\|_F)$ with speed v_n and rate function

$$\mathrm{I} : y \in F \mapsto \inf \{\mathrm{J}(x) : \varphi(x) = y\}$$

iv) If $(W_n)_{n\geq 1}$ is almost surely relatively compact in $(E, \|\cdot\|_E)$ with limit set $K \subset E$, then $(\varphi(W_n))_{n\geq 1}$ is almost surely relatively compact in $(F, \|\cdot\|_F)$ with limit set $\varphi(K)$.

Denote by S_k the bounded linear operator from H to H defined in the basis of eigenvectors of C by :

$$S_k = \sum_{p\neq k} (\lambda_k - \lambda_p)^{-1} \Pi_p, \tag{2}$$

we set

$$\varphi_k : s \in \mathcal{S} \mapsto S_k s \Pi_k + \Pi_k s S_k \in \mathcal{S}, \tag{3}$$

and

$$p_k : s \in \mathcal{S} \mapsto \langle \Pi_k, s\rangle_{\mathcal{S}} \in \mathbb{R}. \tag{4}$$

Note for further references that φ_k and p_k are continuous and linear. Here are our main results.

Theorem 3 *If $(C_n - C)_{n\geq 1}$ converges almost surely to 0 in $\mathcal{S}$, then, for all k,*
i) $(\Pi_{k,n} - \Pi_k)_{n\geq 1}$ converges almost surely to 0 in $\mathcal{S}$.
ii) $(m_{k,n}\lambda_{k,n} - m_k\lambda_k)_{n\geq 1}$ converges almost surely to 0 in $\mathbb{R}$.

Theorem 4 *If for some $b_n \uparrow \infty$, $(b_n (C_n - C))_{n\geq 1}$ converge in law in $\mathcal{S}$ to the limit G_C, then, for all k,*
i) $(b_n (\Pi_{k,n} - \Pi_k))_{n\geq 1}$ converges in law in $\mathcal{S}$ to the limit

$$\mathrm{G}_{\Pi_k} : A \in \mathcal{B}(\mathcal{S}) \mapsto \mathrm{G}_C \left(\varphi_k^{-1}(A)\right).$$

ii) $(b_n (m_{k,n}\lambda_{k,n} - m_k\lambda_k))_{n\geq 1}$ converges in law in $\mathbb{R}$ to the limit

$$\mathrm{G}_{\lambda_k} : B \in \mathcal{B}(\mathbb{R}) \mapsto \mathrm{G}_C \left(p_k^{-1}(B)\right).$$

Theorem 5 *If, for some* $b_n \uparrow \infty$, $\left(b_n\left(C_n - C\right)\right)_{n\geq 1}$ *follows the large deviation principle in* $\mathcal{S}$ *with speed* (v_n) *and rate function* J_C*, then, for all* k*,*
i) $\left(b_n\left(\Pi_{k,n} - \Pi_k\right)\right)_{n\geq 1}$ *follows the large deviation principle in* $\mathcal{S}$ *with rate function*

$$\mathrm{J}_{\Pi_k} : t \in \mathcal{S} \mapsto \inf\left\{\mathrm{J}_C\left(s\right) : \varphi_k\left(s\right) = t\right\}.$$

ii) $\left(b_n\left(m_{k,n}\lambda_{k,n} - m_k\lambda_k\right)\right)_{n\geq 1}$ *follows the large deviation principle in* $\mathbb{R}$ *with rate function*

$$\mathrm{J}_{\lambda_k} : \alpha \in \mathbb{R} \mapsto \inf\left\{\mathrm{J}_C\left(s\right) : p_k\left(s\right) = \alpha\right\}.$$

Theorem 6 *If, for some* $b_n \uparrow \infty$, $\left(b_n\left(C_n - C\right)\right)_{n\geq 1}$ *is almost surely compact in* $\mathcal{S}$ *with limit set* K_C *then, for all* k*,*
i) $\left(b_n\left(\Pi_{k,n} - \Pi_k\right)\right)_{n\geq 1}$ *is almost surely relatively compact in* $\mathcal{S}$ *with limit set*

$$K_{\Pi_k} = \varphi_k\left(K_C\right).$$

ii) $\left(b_n\left(m_{k,n}\lambda_{k,n} - m_k\lambda_k\right)\right)_{n\geq 1}$ *is almost surely relatively compact in* $\mathbb{R}$ *with limit set*

$$K_{\lambda_k} = p_k\left(K_C\right).$$

2 Proofs

In the next lemmas we give some results related to perturbation theory for linear operators useful for our needs.

Definition 7 *Let* Δ *be a self-adjoint element of* $\mathcal{L}\left(H\right)$, λ *be an isolated point of the spectrum of* Δ *we call* Γ *an admissible contour for* λ *and* Δ *whenever* Γ *is a contour around* λ *which contains no other eigenvalues of* Δ*.*

Lemma 8 *i) Let* Δ *be a self-adjoint element of* $\mathcal{L}\left(H\right)$ *and* λ *be an isolated point of the spectrum of* Δ *then for every* Γ*, admissible contour for* λ *and* Δ*, the mapping*

$$\Pi = \frac{1}{2i\pi}\int_{\Gamma}\left(z\mathrm{Id}_H - \Delta\right)^{-1}dz, \tag{5}$$

where $i^2 = -1$*, is the orthogonal projection onto* $\ker\left(\Delta - \lambda\mathrm{Id}_H\right)$*.*
ii) Let Δ *be a self-adjoint element of* $\mathcal{L}\left(H\right)$ *and let* $z \in \mathbb{C}$ *such that* Δ *is not an eigenvalue of* Δ*, then*

$$\left\|\left(z\mathrm{Id}_H - \Delta\right)^{-1}\right\|_{\mathcal{L}} \leq \sup\left\{\left|z - \lambda\right|^{-1} : \lambda \text{ eigenvalue of } \Delta\right\}. \tag{6}$$

Proof. i) See e.g. Proposition 6.3 of [7].
ii) See e.g. Theorem 5.8 of [7]. ■

Lemma 9 *Let Δ be a self-adjoint element of $\mathcal{L}(H)$ with (real) eigenvalues $(l_k)_{k\geq 1}$ of respective multiplicity degrees $(d_k)_{k\geq 1}$ and associated projectors $(P_k)_{k\geq 1}$. Then, for all $k \geq 1$,*

$$\langle P_k, \Delta\rangle_{\mathcal{S}} = d_k l_k.$$

Proof. For all $k \geq 1$, let β_k be an orthonormal basis of the eigen-subspace associated with l_k and let β be any orthonormal basis of H such that $\beta_k \subset \beta$. Note that β_k has d_k elements. Hence, by (1),

$$\begin{aligned}\langle P_k, \Delta\rangle_{\mathcal{S}} &= \sum_{e\in\beta} \langle P_k(e), \Delta(e)\rangle \\ &= \sum_{e\in\beta_k} \langle e, l_k e\rangle = d_k l_k.\end{aligned}$$

■

Set

$$\delta_k = \inf_{p\neq k} |\lambda_k - \lambda_p| = \min(\lambda_k - \lambda_{k+1}, \lambda_{k-1} - \lambda_k).$$

Let Γ_k be the oriented circle with center λ_k and radius $\rho_k = \delta_k/2$. Note that Γ_k is an admissible contour for λ_k and C.
Moreover, define the event

$$\mathcal{O}_{k,n} = \{\|C_n - C\|_{\mathcal{S}} < \delta_k/4\}. \tag{7}$$

Since

$$\sup_{p\in\mathbb{N}} |\lambda_{p,n} - \lambda_p| \leq \|C_n - C\|_{\mathcal{S}} \tag{8}$$

(see e.g. [6] p.99), we can prove :

Lemma 10 *i) For all $\omega \in \mathcal{O}_{k,n}$, Γ_k is an admissible contour for $\lambda_{k,n}(\omega)$ and $C_n(\omega)$.*
ii)

$$\sup_{z\in\Gamma_k} \left\{ \left\| (z\mathrm{Id}_H - C_n)^{-1} \right\|_{\mathcal{L}} \right\} \mathbf{1}_{\mathcal{O}_{k,n}} \leq 4\delta_k^{-1}. \tag{9}$$

iii)

$$\sup_{z\in\Gamma_k} \left\{ \left\| (z\mathrm{Id}_H - C)^{-1} \right\|_{\mathcal{L}} \right\} \leq 2\delta_k^{-1}. \tag{10}$$

Proof. i) Set $\omega \in \mathcal{O}_{k,n}$. By (7) and (8),

$$|\lambda_{k,n}(\omega) - \lambda_k| \leq \delta_k/4 < \rho_k \tag{11}$$

and

$$\begin{aligned}\inf_{p\neq k}\left|\lambda_{p,n}\left(\omega\right)-\lambda_{k}\right| &\geq \inf_{p\neq k}\left|\lambda_{p}-\lambda_{k}\right|-\sup_{p\neq k}\left|\lambda_{p,n}\left(\omega\right)-\lambda_{p}\right| \\ &> \delta_{k}-\delta_{k}/4 \\ &= 3\delta_{k}/4>\rho_{k}. \end{aligned} \tag{12}$$

Hence, the result holds by (11) and (12).
ii) Set $\omega\in\mathcal{O}_{k,n}$. By (6),

$$\sup_{z\in\Gamma_{k}}\left\{\left\|\left(z\mathrm{Id}_{H}-C_{n}\left(\omega\right)\right)^{-1}\right\|_{\mathcal{L}}\right\}\leq\sup\left\{\left|z-\lambda_{p,n}\left(\omega\right)\right|^{-1}:p\in\mathbb{N},z\in\Gamma_{k}\right\}. \tag{13}$$

Moreover, for all $z\in\Gamma_{k}$,

$$\begin{aligned}\left|z-\lambda_{k,n}\left(\omega\right)\right| &\geq \left|z-\lambda_{k}\right|-\left|\lambda_{k}-\lambda_{k,n}\left(\omega\right)\right| \\ &\geq \rho_{k}-\delta_{k}/4=\delta_{k}/4,\end{aligned}$$

and

$$\begin{aligned}\inf_{p\neq k}\left|z-\lambda_{p,n}\left(\omega\right)\right| &= \inf_{p\neq k}\left|\left(\lambda_{p}-\lambda_{k}\right)+\left(\lambda_{k}-z\right)+\left(\lambda_{p,n}\left(\omega\right)-\lambda_{p}\right)\right| \\ &\geq \inf_{p\neq k}\left|\lambda_{p}-\lambda_{k}\right|-\left|\lambda_{k}-z\right|-\sup_{p\neq k}\left|\lambda_{p,n}\left(\omega\right)-\lambda_{p}\right| \\ &\geq \delta_{k}-\rho_{k}-\delta_{k}/4=\delta_{k}/4.\end{aligned}$$

Therefore,

$$\inf_{z\in\Gamma_{k}}\inf_{p\in\mathbb{N}}\left|z-\lambda_{p,n}\left(\omega\right)\right|\geq\delta_{k}/4,$$

which, combined with (13), give the result.
iii) Note that, for all $z\in\Gamma_{k}$,

$$\left|\lambda_{k}-z\right|=\delta_{k}/2,$$

and

$$\begin{aligned}\inf_{p\neq k}\left|\lambda_{p}-z\right| &\geq \inf_{p\neq k}\left|\lambda_{p}-\lambda_{k}\right|-\left|\lambda_{k}-z\right| \\ &\geq \delta_{k}-\delta_{k}/2=\delta_{k}/2.\end{aligned}$$

Therefore, using (6), we get,

$$\begin{aligned}\sup_{z\in\Gamma_{k}}\left\{\left\|\left(z\mathrm{Id}_{H}-C\right)^{-1}\right\|_{\mathcal{L}}\right\} &\leq \sup\left\{\left|z-\lambda_{p}\right|^{-1}:p\in\mathbb{N},z\in\Gamma_{k}\right\} \\ &\leq 2\delta_{k}^{-1}.\end{aligned}$$

■
Now, we can state the main tools used in the proof of our theorems.

Proposition 11 *For all k,*
i) There exists a $\mathcal{S}$-valued random variable $R_{k,n}$ such that, for every $n \geq 1$,

$$\Pi_{k,n} - \Pi_k = \varphi_k (C_n - C) + R_{k,n}, \tag{14}$$

and

$$\|R_{k,n}\|_{\mathcal{S}} \mathbf{1}_{\mathcal{O}_{k,n}} \leq 8\delta_k^{-2} \|C_n - C\|_{\mathcal{S}}^2 . \tag{15}$$

ii) There exists a real valued random variable $r_{k,n}$ such that, for every $n \geq 1$,

$$m_{k,n}\lambda_{k,n} - m_k\lambda_k = p_k (C_n - C) + r_{k,n}, \tag{16}$$

and, for some $\tau_k > 0$ and $\gamma_k > 0$,

$$|r_{k,n}| \mathbf{1}_{\mathcal{O}_{k,n}} \leq \tau_k \|C_n - C\|_{\mathcal{S}}^2 + \gamma_k \|C_n - C\|_{\mathcal{S}}^3 . \tag{17}$$

Proof. i) Set $\omega \in \mathcal{O}_{k,n}$. Since, by the first part of Lemma 10, Γ_k is an admissible contour for λ_k and $C_n(\omega)$ (and also for C), (6) implies that

$$\Pi_{k,n}(\omega) - \Pi_k = \frac{1}{2i\pi} \int_{\Gamma_k} (z\mathrm{Id}_H - C_n(\omega))^{-1} - (z\mathrm{Id}_H - C)^{-1} dz. \tag{18}$$

For convenience, set

$$a = z\mathrm{Id}_H - C_n(\omega) \quad \text{and} \quad b = z\mathrm{Id}_H - C.$$

Note that

$$\begin{aligned} a^{-1} - b^{-1} &= a^{-1}(b-a)b^{-1} \\ &= b^{-1}(b-a)b^{-1} + \left(a^{-1} - b^{-1}\right)(b-a)b^{-1} \\ &= b^{-1}(b-a)b^{-1} + b^{-1}(b-a)a^{-1}(b-a)b^{-1}. \end{aligned} \tag{19}$$

Therefore, if we set

$$U_{k,n} = \frac{\mathbf{1}_{\mathcal{O}_{k,n}}}{2i\pi} \int_{\Gamma_k} (z\mathrm{Id}_H - C)^{-1} (C_n - C) (z\mathrm{Id}_H - C_n)^{-1} (C_n - C) (z\mathrm{Id}_H - C)^{-1} dz,$$

we get, by (18) and (19),

$$\Pi_{k,n}(\omega) - \Pi_k = \frac{1}{2i\pi} \left(\int_{\Gamma_k} (z\mathrm{Id}_H - C)^{-1} (C_n(\omega) - C) (z\mathrm{Id}_H - C)^{-1} dz \right) + U_{k,n}(\omega).$$

Now, in [3] p.145, it is shown that

$$\varphi_k : s \in \mathcal{S} \mapsto \frac{1}{2i\pi} \left(\int_{\Gamma_k} (z\mathrm{Id}_H - C)^{-1} s (z\mathrm{Id}_H - C)^{-1} dz \right).$$

Hence, if we define

$$R_{k,n} = U_{k,n} + (\Pi_{k,n} - \Pi_k - \varphi_k (C_n - C)) \mathbf{1}_{\mathcal{O}_{k,n}^c},$$

(14) holds. Moreover, following [3] p.142 (lines 2 and 3), we obtain, using (9) and (10), that

$$\begin{aligned}
& \|U_{k,n}\|_{\mathcal{S}} \\
\leq \;& \rho_k \sup_{z\in\Gamma_k} \left(\left\| (z\mathrm{Id}_H - C)^{-1} (C_n - C) (z\mathrm{Id}_H - C_n)^{-1} (C_n - C) (z\mathrm{Id}_H - C)^{-1} \right\|_{\mathcal{S}} \right) \mathbf{1}_{\mathcal{O}_{k,n}} \\
\leq \;& \frac{\delta_k}{2} \|C_n - C\|_{\mathcal{S}}^2 \sup_{z\in\Gamma_k} \left\{ \left\| (z\mathrm{Id}_H - C)^{-1} \right\|_{\mathcal{L}}^2 \left\| (z\mathrm{Id}_H - C_n)^{-1} \right\|_{\mathcal{L}} \right\} \mathbf{1}_{\mathcal{O}_{k,n}} \\
\leq \;& 8\delta_k^{-2} \|C_n - C\|_{\mathcal{S}}^2 .
\end{aligned}$$

ii) Observe that, by lemma 9,

$$\begin{aligned}
& m_{k,n}\lambda_{k,n} - m_{k,n}\lambda_k \\
= \;& \langle \Pi_{k,n}, C_n \rangle_{\mathcal{S}} - \langle \Pi_k, C \rangle_{\mathcal{S}} \\
= \;& \langle \Pi_k, C_n - C \rangle_{\mathcal{S}} + \langle \Pi_{k,n} - \Pi_k, C_n \rangle_{\mathcal{S}} \\
= \;& \langle \Pi_k, C_n - C \rangle_{\mathcal{S}} + \langle \Pi_{k,n} - \Pi_k, C \rangle_{\mathcal{S}} + \langle \Pi_{k,n} - \Pi_k, C_n - C \rangle_{\mathcal{S}} \\
= \;& p_k (C_n - C) + \langle \varphi_k (C_n - C), C \rangle_{\mathcal{S}} \qquad (20) \\
& + \langle R_{k,n}, C \rangle_{\mathcal{S}} + \langle \Pi_{k,n} - \Pi_k, C_n - C \rangle_{\mathcal{S}} .
\end{aligned}$$

Furthermore, let $(e_p)_{p\geq 1}$ be an orthonormal basis of H such that e_k is an eigenvector of C associated with λ_k. Then, by (1), (3) and (2), for all $s \in \mathcal{S}$,

$$\begin{aligned}
& \langle \varphi_k (s), C \rangle_{\mathcal{S}} \\
= \;& \sum_p \langle S_k s \Pi_k (e_p), C(e_p) \rangle + \sum_p \langle \Pi_k s S_k (e_p), C(e_p) \rangle \\
= \;& \lambda_k \langle S_k s (e_k), e_k \rangle + \sum_{p\neq k} \frac{\lambda_p}{\lambda_k - \lambda_p} \langle \Pi_k s (e_p), e_p \rangle \\
= \;& 0. \qquad (21)
\end{aligned}$$

Hence, if we combine (20) and (21), we get

$$m_{k,n}\lambda_{k,n} - m_k\lambda_k = p_k (C_n - C) + r_{k,n},$$

where

$$r_{k,n} = \langle R_{k,n}, C \rangle_{\mathcal{S}} + \langle \Pi_{k,n} - \Pi_k, C_n - C \rangle_{\mathcal{S}}$$

satisfies

$$\begin{aligned}
|r_{k,n}| \mathbf{1}_{\mathcal{O}_{k,n}} \;&\leq\; \|C\|_{\mathcal{S}} \|R_{k,n}\|_{\mathcal{S}} \mathbf{1}_{\mathcal{O}_{k,n}} + \|C_n - C\|_{\mathcal{S}} \|\Pi_{k,n} - \Pi_k\|_{\mathcal{S}} \mathbf{1}_{\mathcal{O}_{k,n}} \\
&\leq\; \left(8\delta_k^{-2} \|C\|_{\mathcal{S}} + \|\varphi_k\|_{\mathcal{L}(\mathcal{S})} \right) \|C_n - C\|_{\mathcal{S}}^2 + 8\delta_k^{-2} \|C_n - C\|_{\mathcal{S}}^3 ,
\end{aligned}$$

where $\|\cdot\|_{\mathcal{L}(\mathcal{S})}$ is the usual norm on the space $\mathcal{L}(\mathcal{S})$ of bounded linear $\mathcal{S}$-valued operators. ■

Remark 12 *It is clear that the proof of* (14) *may be easily adapted when the norm* $\|\cdot\|_{\mathcal{S}}$ *is replaced by the weaker norm* $\|\cdot\|_{\mathcal{L}}$. *Hence, if* C_n *converges to* C *only in* $\mathcal{L}$, *all the conclusions of theorems 3, 4, 5 and 6 remains valid with the norm* $\|\cdot\|_{\mathcal{L}}$ *for the projectors* $\Pi_{k,n} - \Pi_k$. *On the opposite, this is not the case for the eigen-values since the proof of* (16) *depends strongly of the Hilbert-Schmidt norm via Lemma 9.*

Remark 13 *Note also that* (14) *and* (16) *may be used to get informations on the global behaviour of the random variables*

$$\{\Pi_{k,n} - \Pi_k : k \geq 1\} \quad \textit{and} \quad \{m_{k,n}\lambda_{k,n} - m_k\lambda_k : k \geq 1\}.$$

In particular,

$$\max_{k \leq k_n} \|\Pi_{k,n} - \Pi_k\|_{\mathcal{S}} \quad \textit{and} \quad \max_{k \leq k_n} |m_{k,n}\lambda_{k,n} - m_k\lambda_k|,$$

can be studied, where (k_n) *is a well-chosen sequence increasing to* ∞ *(see Chapter 4 of Bosq [2] for related applications). This will be done elsewhere.*

Using Lemma 11, the proofs of our theorems are now simple exercises.

Proof of Theorem 3 :

i) The law of large numbers for $(C_n - C)$, the continuity of φ_k and (15) give,

$$\limsup_{n\to\infty} \left\|\varphi_k (C_n - C) + R_{k,n}\mathbf{1}_{\mathcal{O}_{k,n}}\right\|_{\mathcal{S}} = 0 \text{ a.s.}$$

Hence, by (14), we just need to show that

$$\limsup_{n\to\infty} \left\|R_{k,n}\mathbf{1}_{\mathcal{O}_{k,n}^c}\right\|_{\mathcal{S}} = 0 \text{ a.s.},$$

which is obvious since for all

$$\omega \in \left\{\limsup_{n\to\infty} \|C_n - C\|_{\mathcal{S}} = 0\right\}$$

and all large n,

$$\mathbf{1}_{\mathcal{O}_{k,n}^c}(\omega) = \mathbf{1}_{\left\{\|C_n(\omega) - C\|_{\mathcal{S}} \geq \delta_k/4\right\}} = 0. \tag{22}$$

ii) The law of large numbers for $(C_n - C)$, the continuity of p_k and (17) give,

$$\limsup_{n\to\infty} \left|p_k (C_n - C) + r_{k,n}\mathbf{1}_{\mathcal{O}_{k,n}}\right| = 0 \text{ a.s.}$$

Hence, by (14), we just need to show that

$$\limsup_{n\to\infty} \left| r_{k,n} \mathbf{1}_{\mathcal{O}_{k,n}^c} \right| = 0 \text{ a.s.}$$

which is clear using (22). ■

Proof of Theorem 4 :

i) The linearity of φ_k and (14) entail

$$b_n\left(\Pi_{k,n} - \Pi_k\right) = \varphi_k\left[b_n\left(C_n - C\right)\right] + b_n R_{k,n}. \tag{23}$$

Therefore, by Lemma 2 and Theorem 4.1 of Bilingsley [1], we just have to show that, for all $\eta > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(b_n \left\|R_{k,n}\right\|_{\mathcal{S}} \geq \eta\right) = 0.$$

To this aim, observe that, for all $\alpha > 0$, (15) leads to, for n large enough,

$$\begin{aligned}
& \mathbb{P}\left(b_n \left\|R_{k,n}\right\|_{\mathcal{S}} \geq \eta\right) \\
\leq\ & \mathbb{P}\left(b_n \left\|C_n - C\right\|_{\mathcal{S}}^2 \geq \frac{\eta\delta_k^2}{8}\right) + \mathbb{P}\left(\mathcal{O}_{k,n}^c\right) \\
\leq\ & \mathbb{P}\left(b_n \left\|C_n - C\right\|_{\mathcal{S}} \geq \left(\frac{\eta\delta_k^2 b_n}{8}\right)^{1/2}\right) + \mathbb{P}\left(b_n \left\|C_n - C\right\|_{\mathcal{S}} \geq \frac{\delta_k}{4} b_n\right) \\
\leq\ & 2\,\mathbb{P}\left(b_n \left\|C_n - C\right\|_{\mathcal{S}} \geq \alpha\right).
\end{aligned} \tag{24}$$

Hence, using the convergence in law of $b_n\left(C_n - C\right)$ we get

$$\begin{aligned}
\limsup_{n\to\infty} \mathbb{P}\left(b_n \left\|R_{k,n}\right\|_{\mathcal{S}} \geq \eta\right) &\leq 2 \limsup_{\alpha\to\infty} \limsup_{n\to\infty} \mathbb{P}\left(b_n \left\|C_n - C\right\|_{\mathcal{S}} \geq \alpha\right) \\
&\leq 2 \limsup_{\alpha\to\infty} \mathrm{G}_C\left(\left\{s \in \mathcal{S} : \left\|s\right\|_{\mathcal{S}} \geq \alpha\right\}\right) \\
&= 0.
\end{aligned}$$

ii) The linearity of p_k and (16) give

$$b_n\left(m_{k,n}\lambda_{k,n} - m_k\lambda_k\right) = p_k\left[b_n\left(C_n - C\right)\right] + b_n r_{k,n}. \tag{25}$$

Therefore, by Lemma 2 and Theorem 4.1 of Billingsley [1], we just have to show that, for all $\eta > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(b_n \left|r_{k,n}\right| \geq \eta\right) = 0.$$

But, for all $\alpha > 0$, (15) entails, for all large n,

$$\begin{aligned}
& \mathbb{P}\left(b_n \left|r_{k,n}\right| \geq \eta\right) \\
\leq\ & \mathbb{P}\left(b_n^2 \left\|C_n - C\right\|_{\mathcal{S}}^2 \geq \tau_k^{-1} \frac{\eta}{2} b_n\right) \\
& + \mathbb{P}\left(b_n^3 \left\|C_n - C\right\|_{\mathcal{S}}^3 \geq \gamma_k^{-1} \frac{\eta}{2} b_n^2\right) + \mathbb{P}\left(\mathcal{O}_{k,n}^c\right) \\
\leq\ & 3\,\mathbb{P}\left(b_n \left\|C_n - C\right\|_{\mathcal{S}} \geq \alpha\right).
\end{aligned} \tag{26}$$

Hence,

$$\begin{aligned}\limsup_{n\to\infty}\mathbb{P}\left(b_n\left|r_{k,n}\right|\geq\eta\right) &\leq 3\limsup_{\alpha\to\infty}\limsup_{n\to\infty}\mathbb{P}\left(b_n\left\|C_n-C\right\|_{\mathcal{S}}\geq\alpha\right)\\ &= 0.\end{aligned}$$

■

Proof of Theorem 5 :

i) By (23), Lemma 2 and Theorem 4.2.13 of Dembo and Zeitouni [4], we just have to show that, for all $\eta>0$,

$$\limsup_{n\to\infty}v_n\log\mathbb{P}\left(b_n\left\|R_{k,n}\right\|_{\mathcal{S}}\geq\eta\right)=-\infty.$$

But, (24) and the large deviation principle of $b_n\left(C_n-C\right)$ give

$$\begin{aligned}&\limsup_{n\to\infty}v_n\log\mathbb{P}\left(b_n\left\|R_{k,n}\right\|_{\mathcal{S}}\geq\eta\right)\\ \leq\ &\limsup_{\alpha\to\infty}\limsup_{n\to\infty}v_n\log\mathbb{P}\left(b_n\left\|C_n-C\right\|_{\mathcal{S}}\geq\alpha\right)\\ \leq\ &\limsup_{\alpha\to\infty}-\inf\left\{\mathrm{J}_C\left(s\right):\left\|s\right\|_{\mathcal{S}}\geq\alpha\right\}=-\infty.\end{aligned}$$

ii) Using (25), Lemma 2 and Theorem 4.2.13 of Dembo and Zeitouni [4], we just have to show that, for all $\eta>0$,

$$\limsup_{n\to\infty}v_n\log\mathbb{P}\left(b_n\left|r_{k,n}\right|\geq\eta\right)=-\infty.$$

But, (26) and the large deviation principle of $b_n\left(C_n-C\right)$ give

$$\begin{aligned}&\limsup_{n\to\infty}v_n\log\mathbb{P}\left(b_n\left|r_{k,n}\right|\geq\eta\right)\\ \leq\ &\limsup_{\alpha\to\infty}\limsup_{n\to\infty}v_n\log\mathbb{P}\left(b_n\left\|C_n-C\right\|_{\mathcal{S}}\geq\alpha\right)\\ \leq\ &\limsup_{\alpha\to\infty}-\inf\left\{\mathrm{J}_C\left(s\right):\left\|s\right\|_{\mathcal{S}}\geq\alpha\right\}=-\infty.\end{aligned}$$

■

Proof of Theorem 6 :

i) By (23) and Lemma 2, it is enough to show that

$$\limsup_{n\to\infty}b_n\left\|R_{k,n}\right\|_{\mathcal{S}}=0\text{ a.s.}$$

Now, since $\left(b_n\left(C_n-C\right)\right)$ is almost surely relatively compact

$$\begin{aligned}\limsup_{n\to\infty}b_n\left\|C_n-C\right\|_{\mathcal{S}} &= \sup\left\{\left\|s\right\|_{\mathcal{S}}:s\in K\right\}\ \text{a.s.}\\ &< \infty\ \text{a.s.}\end{aligned}\tag{27}$$

Hence,

$$\limsup_{n\to\infty} \|C_n - C\|_{\mathcal{S}} = 0 \text{ a.s.} \tag{28}$$

and

$$\limsup_{n\to\infty} b_n \|C_n - C\|^2_{\mathcal{S}} = 0 \text{ a.s.} \tag{29}$$

using (15) and (22), we get

$$\begin{aligned} \limsup_{n\to\infty} b_n \|R_{k,n}\|_{\mathcal{S}} &\leq 8\delta_k^{-2} \limsup_{n\to\infty} b_n \|C_n - C\|^2_{\mathcal{S}} \\ &\quad + \limsup_{n\to\infty} b_n \|R_{k,n}\|_{\mathcal{S}} \mathbf{1}_{\mathcal{O}^c_{k,n}} \\ &= 0 \text{ a.s.} \end{aligned}$$

ii) By (25) and Lemma 2, we just have to show that

$$\limsup_{n\to\infty} b_n |r_{k,n}| = 0 \text{ a.s.}$$

But (17) and (27), we get

$$\begin{aligned} \limsup_{n\to\infty} b_n |r_{k,n}| &\leq \tau_k \limsup_{n\to\infty} b_n \|C_n - C\|^2_{\mathcal{S}} + \gamma_k \limsup_{n\to\infty} b_n \|C_n - C\|^3_{\mathcal{S}} \\ &\quad + \limsup_{n\to\infty} b_n \|R_{k,n}\|_{\mathcal{S}} \mathbf{1}_{\mathcal{O}^c_{k,n}} \\ &= 0 \text{ a.s.} \end{aligned}$$

■

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