INSTITUT NATIONAL DE LA STATISTIQUE ET DES ETUDES ECONOMIQUES
Série des Documents de Travail du CREST
(Centre de Recherche en Economie et Statistique)

n° 2001-20

Compound Autoregressive Models

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June, 2001

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Abstract

We introduce a class of nonlinear dynamic processes, called compound autoregressive (CAR), and characterized by the conditional log-Laplace transforms which are affine functions of the lagged values of the process. The CAR processes resemble the linear autoregressive processes in that their forecasting distributions at all horizons admit analytical expressions, and can be used to derive the ergodicity conditions, the stationary distribution, as well as the filtering formula. Among the CAR processes we are able to identify and describe all reversible processes, i.e. those with identical distributional properties in calendar and reverse time. In particular, we study their nonlinear canonical decompositions that arise as extensions of nonlinear canonical decompositions of a gaussian AR(1) process and the Cox-Ingersoll-Ross model. Estimation of CAR models can be performed using either parametric or nonparametric methods. For illustration, a *CAR* model is fitted to a series of financial returns, and its application to nonparametric derivative pricing is also discussed.

Keywords: Nonlinear Dynamics, Laplace Transform, Nonlinear Canonical Analysis, Cox-Ingersoll-Ross Process, Kernel Estimation, Derivative Pricing.

JEL Classification: C14, C22.

Résumé

Nous introduisons une classe de modèles dynamiques non linéaires, dont les propriétés sont similaires à celles d'un processus autorégressif gaussien. Les modèles autorégressifs composés (*CAR*) sont définis à partir du logarithme de la transformée de Laplace conditionnelle, supposée affine en la variable conditionnante. Nous donnons la forme nécessaire de la distribution invariante, la condition d'ergodicité, la décomposition canonique non linéaire, et nous décrivons tous les processus *CAR* réversibles. Ce type de modèle est enfin appliqué à une série de rendements financiers et à la valorisation non paramétrique de produits dérivés.

Mots clés: Dynamique non linéaire, Transformée de Laplace, Décomposition canonique non linéaire, Processus de Cox-Ingersoll-Ross, Estimation par Noyau, Valorisation de produits dérivés.

Classification JEL: C14, C22.

1 Introduction

The aim of this paper is to introduce nonlinear dynamic models, called compound autoregressive processes (CAR), with the same type of properties as a gaussian autoregressive model :

$$y_t = \rho y_{t-1} + \varepsilon_t, \varepsilon_t \ IIN(0, \sigma^2).$$

Especially the gaussian dynamics lends itself easily to prediction making. For instance the linear prediction at any horizon is given by : $E(y_{t+h}|y_t) = \rho^h y_t$, that is depends on the power of the autoregressive coefficient. The nonlinear predictions are also easily computed since :

$$E[H_j(y_{t+1})|y_t] = \rho^j H_j(y_t), j \text{ varying},$$

where H_j denotes the Hermite polynomial of degree j. This property allows us to compute the prediction of any nonlinear transformation of y by expanding this transformation on the basis of Hermite polynomials. Finally, in a multivariate linear setup, it is also easy to derive the prediction of latent factors by using the Kalman filter.

The conditionally gaussian models are also important in financial applications. This is due to the simple expression of the conditional moment generating function (or Laplace transform) :

$$E[\exp -uy_t|y_{t-1}] = \exp[-u\rho y_{t-1} + \frac{u^2\sigma^2}{2}].$$

For instance, a portfolio management based on an expected Constant Absolute Risk Aversion (CARA) utility function is equivalent to a simple mean-variance management. Moreover, for derivative pricing, explicit expressions of risk neutral distributions are easily derived from the Girsanov theorem. Typical examples are the Black-Scholes and Vasicek models.

However, in applied finance there is a pressing need for nonlinear dynamic models that are easily applicable to both prediction making and derivative pricing. Nonlinear features to be accommodated arise from the specific problems of interest, such as forecasting at various horizons, value constrained variables, as for instance, interest rates, volatilities, intertrade durations, or trade counts, that admit only nonnegative or discrete values. In derivative pricing, nonlinearity arises from the form of dependence between the cash-flow and price of underlying asset.

There exist three different approaches that allow to specify a nonlinear relation between Y_t and Y_{t-1}, say.

i) The joint distribution can be defined by the joint cumulative distribution function (cdf henceforth) $F(y_t, y_{t-1})$, which can be written as:

$$F(y_t, y_{t-1}) = C[F_1(y_t), F_2(y_{t-1})],$$

where F_1 (resp. F_2) is the marginal cdf of Y_t (resp. Y_{t-1}), and C is a copula, that is a cdf whose marginal distributions are uniform on the interval [0,1], [see e.g. Joe (1997)].

ii) The joint distribution can also be analyzed through the joint probability density function (pdf henceforth) $f(y_t, y_{t-1})$ and its nonlinear canonical decomposition:

$$f(y_t, y_{t-1}) = f_1(y_t) f_2(y_{t-1}) \left\{ 1 + \sum_{n=1}^{\infty} \lambda_n \Phi_n(y_t) \Psi_n(y_{t-1}) \right\},$$

where f_1 (resp. f_2) is the marginal pdf of Y_t (resp. Y_{t-1}), λ_n, $0 \leq \lambda_n \leq 1$, is the decreasing sequence of nonlinear canonical correlations and Φ_n (resp. Ψ_n) are the so-called current [resp. lagged] canonical variates. This decomposition allows to distinguish the marginal effects in f_1, f_2 from the time dependence effects in $\lambda_n, \Phi_n, \Psi_n$.

iii) Finally, the joint distribution can be characterized by the joint Laplace transform [or moment generating function], defined as $E \exp(uY_t + vY_{t-1})$, for any u, v.

The latter approach seems better suited for financial applications due to important role of the exponential function in finance. This function appears, for instance, in a) the formula of the CARA utility function, b) the definition of a european call with a cash-flow that is a simple function of an exponential transform of geometric returns [1] c) standard specifications of the stochastic discount factors used in derivative pricing.

This paper develops an approach based on the Laplace transform. We introduce a class of nonlinear dynamic models, that are similar to gaussian autoregressive processes with respect to their properties. We call them compound autoregressive (CAR) processes . A CAR model is defined in Section 2 in terms of the conditional log-Laplace transform, which is an affine function of the past values of the process. We give the necessary form of the invariant distribution of a CAR, and we explain how to perform predictions at any horizon h. In Section 3 we consider the nonlinear prediction problem. We examine spectral decomposition of the conditional expectation operator. We show that the eigenvalues form a geometric sequence while the associated eigenfunctions are polynomials of increasing orders. The time reversibility condition is discussed in Section 4. In particular, we describe various reversible compound autoregressive models, such as the autoregressive gaussian, the autoregressive gamma, and the compound Poisson processes. A CAR process admits a nonlinear state space representation. The corresponding filtering and smoothing equations are presented in Section 5. In Section 6 we consider multivariate extensions of CAR processes and we

[1] Note that a european call written on a price S_{t+1} with payoff $(S_{t+1} - kS_t)^+$, where k is the moneyness strike, is a multiple of the call written on $\exp y_{t+1}$ with payoff $(S_{t+1}/S_t - k)^+ = (\exp y_{t+1} - k)^+$, where y_{t+1} is the geometric return of the underlying asset.

derive the associated stationarity condition. Nonparametric statistical inference is discussed in Section 7 and the *CAR* specification is estimated on a series of financial returns on Standard and Poor 500 in Section 8. Finally, we discuss the nonparametric derivative asset pricing in Section 9, and we conclude in Section 10.

2 The Autoregressive Specification

In this section the dynamics of the process is defined by a Laplace transform with autoregressive path dependence. For this reason, the model under study shares some common features with a standard gaussian AR(1) process. First, we define the *CAR* process and introduce the forecasting formula. Next we derive the ergodicity conditions.

2.1 Definition

Let us consider a n-dimensional process $Y = (Y_t, \, t \geq 0)$ and denote by $\underline{Y_{t-1}}$ the information set including the lagged values of the process up to and including $t-1$.

Definition 2.1 *: The process Y is compound autoregressive of order p [CAR(p)] if and only if the conditional distribution of Y_t given $\underline{Y_{t-1}}$ admits a conditional Laplace transform of the type:*

$$E\left[\exp(-u'Y_t) \mid \underline{Y_{t-1}}\right] = \exp\left[-a_1'(u)\, Y_{t-1} - \ldots - a_p'(u)\, Y_{t-p} + b(u)\right], \tag{2.1}$$

where $a_p \neq 0$.

Thus we assume that the Log-Laplace transform of the conditional distribution is an affine function of the p most recent lagged values of the process. In particular Y is a Markov process of order p. The Laplace transform may not be defined for all values $u \in \mathbb{R}^n$. It is known that in one dimensional processes, it is defined on an interval including zero. The size of this interval depends on the tails of the conditional distribution. In particular, its lower bound is zero, when the decay rate of the right tail is less than exponential, and it is infinite for a thin right tail. The same result holds for the upper bound of the interval, determined by the left tail. In the sequel we will assume that the Laplace transform is defined in an open neighbourhood of zero [2]. Under this assumption the Laplace transform characterizes the distribution.

[2] We will not attempt to estimate the support of the Laplace transform, which would be equivalent to examining the tails. Note that the conditional tails of financial returns are generally thinner than or equal to exponential tails.

Like in the class of gaussian autoregressive processes, a study of a CAR process of order p is equivalent to examining a process of order one obtained by stacking the lagged values into a vector of dimension p.

Proposition 2.1 *: The process Y is a CAR(p) process if and only if the process $(\widetilde{Y}_t) = (Y_t', Y_{t-1}', ..., Y_{t-p}')'$ is a CAR(1) process.*

Proof: Let us consider a *CAR(p)* process. The conditional Laplace transform associated with the conditional p.d.f. of $\widetilde{Y}$ is:

$$
\begin{aligned}
& E\left[\exp(-v'\widetilde{Y}_t) \mid \underline{\widetilde{Y}_{t-1}}\right] \\
= \; & E\left[\exp(-v_1'Y_t - v_2'Y_{t-1} - ... - v_p'Y_{t-p+1}) \mid \underline{Y_{t-1}}\right] \\
= \; & E\left[\exp(-v_1'Y_t) \mid \underline{Y_{t-1}}\right] \times \exp(-v_2'Y_{t-1} - ... - v_p'Y_{t-p+1}) \\
= \; & \exp\left(-[a_1'(v_1) + v_2']Y_{t-1} - ... - [a_{p-1}'(v_1) + v_p']Y_{t-p+1} - a_p'(v_1)Y_{t-p} + b(v_1)\right)
\end{aligned}
$$

QED

The above expression of the Laplace Transform defines a $CAR(1)$ process with:

$$
a(v) = \begin{bmatrix} a_1(v_1) + v_2 \\ . \\ . \\ a_{p-1}(v_1) + v_p \\ a_p(v_1) \end{bmatrix}.
$$

2.2 Examples

In the definition of the distribution in terms of its Laplace transform, we do not distinguish the set of admissible values of the process. The process can be continuously valued as well as discretely valued. Various examples described below depend on the set of admissible values of the process Y. For clarity of exposition, we consider univariate processes of order one.

Example 2.1 *:* **Autoregressive gaussian process**
When: $Y_t = \rho Y_{t-1} + \varepsilon_t$, where (ε_t) is a standard gaussian white noise, we get:

$$
E\left[\exp(-uY_t) \mid Y_{t-1}\right] = \exp\left[-u\rho Y_{t-1} + \frac{u^2}{2}\right],
$$

and $a(u) = \rho u$, $b(u) = \frac{u^2}{2}$.

Example 2.2 *: **Integer valued process***

This type of processes is useful in the queuing analysis. Let us assume that Y_{t-1} counts the number of individuals in a queue at the end of period $t-1$, or equivalently at the beginning of period t. Let us denote by ε_t the number of individuals arriving in the queue at period t and by Z_t the number of individuals among the waiting Y_{t-1} individuals who have not been served at this period. We get :

$$Y_t = Z_t + \varepsilon_t.$$

It is not possible to specify a standard linear autoregression :

$$Y_t = \rho Y_{t-1} + \varepsilon_t,$$

since $Z_t = \rho Y_{t-1}$ has to take integer values. However the deterministic autoregression can be replaced by a stochastic autoregression, where the individuals are randomly served with probability ρ :

$$Y_t = \sum_{i=1}^{Y_{t-1}} Z_{i,t} + \varepsilon_t,$$

where the variables $Z_{i,t}$ admit Bernoulli distribution $\mathcal{B}(1,\rho)$

More generally let us introduce integer valued independent variables $Z_{i,t}$, i , $t \in N$, and ε_t, t varying. We assume that the Z variables admit a Laplace transform:

$$E\left[\exp(-uZ)\right] = \exp\left[-a(u)\right],$$

whereas the Laplace transform of ε is:

$$E\left[\exp(-u\varepsilon)\right] = \exp\left[b(u)\right].$$

Then the process defined by:

$$Y_t = \sum_{i=1}^{Y_{t-1}} Z_{i,t} + \varepsilon_t,$$

admits the conditional Laplace transform:

$$E\left[\exp(-u'Y_t) \mid \underline{Y_{t-1}}\right] = \exp\left[-a\left(u\right)Y_{t-1} + b\left(u\right)\right].$$

Y_t arises as the sum of a random number of variables Z_i; this is why the process is called a "compound process"[3].

Example 2.3 *: **Nonnegative valued process***

Similarly, in order to build a CAR process with nonnegative real values, we set:

[3]It is called thinning model in Grunwald et alii (1997)

- $\exp[b(u)]$, *the Laplace transform of a nonnegative variable* ϵ_t*;*
- $\exp[-a(u)]$, *the Laplace transform of another nonnegative variable, with an infinitely divisible distribution.*

The latter example shows that the class of CAR models is quite large. Indeed, we see that to characterize the Laplace transform of a nonnegative variable, we can choose for function b any infinitely differentiable function on $\mathbb{R}^+$ *such that* $b(0) = 0$ *and* $\exp b(u)$ *satisfies the property of complete monotonicity (see [18, Feller (1971)]):*

$$\forall j, u : (-1)^j \frac{d^j}{du^j}[\exp b(u)] \geq 0.$$

Similarly we can select for function a any infinitely differentiable function of $\mathbb{R}^+$ *such that* $a(0) = 0$, $\exp[-a(u)]$ *satisfies the property of complete monotonicity, and* $a(u)$ *is an increasing function with alternating signs of the derivatives (see [27, Joe (1997), Theorem A.1]).*

2.3 Invariant distributions

Since a *CAR(1)* process is a Markov process of order one, we are interested in finding the invariant distributions of Y with Laplace transform:

$$E\left[\exp(-u'Y_t)\right] = \exp\left[c(u)\right]. \tag{2.2}$$

By the invariance property, we get:

$$\begin{aligned}\exp\left[c(u)\right] &= E\left[\exp(-u'Y_t)\right] = E\left[E\left[\exp(-u'Y_t) \mid Y_{t-1}\right]\right] \\ &= E\left[\exp\left[-a'(u)\,Y_{t-1} + b(u)\right]\right] \\ &= \exp\left[c[a(u)] + b(u)\right].\end{aligned}$$

Proposition 2.2 *: The log-Laplace transform of an invariant distribution of a CAR(1) process is a function c such that* [4]*:*

$$b(u) = c(u) - c[a(u)].$$

Example 2.4 *: For a gaussian autoregressive process, it is known that the equation in Proposition 2.2 admits a unique solution* $c(u) = u^2/[2(1-\rho^2)]$ *corresponding to the gaussian distribution* $N[1,\ 1/(1-\rho^2)]$*, if* $|\rho| \neq 1$*, and has no solution in the unit root case.*

[4] Note that the c function satisfying this condition does not necessarily exist, and if it exists it is not necessarily unique.

Therefore, if an invariant distribution exists, we can parametrize the conditional Laplace transform either by a and b, or by a and c. In the first case the functional parameters a and b represent the time dependence and distribution of the innovations [see examples 2.2, 2.3]; in the second case a and c represent the time dependence and marginal distribution, respectively. Under the latter parametrization, we get:

$$E\left[\exp(-u'Y_t) \mid Y_{t-1}\right] = \exp\left[-a'(u)\, Y_{t-1} + c(u) - c[a(u)]\right]. \qquad (2.3)$$

2.4 Forecasts at various horizons

We derive the forecasting distribution [this approach is called density forecasting] at any horizon h for a *CAR(1)* process.

Proposition 2.3 *: For a CAR(1) process we get:*

$$E\left[\exp(-u'Y_t) \mid \underline{Y_{t-h}}\right] = \exp\left[-a^{\circ h}(u)'Y_{t-h} + \sum_{k=0}^{h-1} b[a^{\circ k}(u)]\right], \qquad (2.4)$$

where $a^{\circ h}$ denotes the function a compounded h times with itself.

Proof: The result is obtained by recursion •

The forecasting formula simplifies, when the process admits a stationary distribution associated with the Log-Laplace transform c. Indeed we get:

$$\sum_{k=0}^{h-1} b[a^{\circ k}(u)] = \sum_{k=0}^{h-1} \left\{c[a^{\circ k}(u)] - c[a^{\circ (k+1)}(u)]\right\} = c(u) - c[a^{\circ h}(u)].$$

Corollary 2.1 *: For a CAR(1) process with an invariant Log-Laplace transform c, we get:*

$$E\left[\exp(-u'Y_t) \mid \underline{Y_{t-h}}\right] = \exp\left[-a^{\circ h}(u)'Y_{t-h} + c(u) - c[a^{\circ h}(u)]\right]. \qquad (2.5)$$

We deduce an ergodicity condition for the process Y.

Corollary 2.2 *: Let us consider a CAR(1) process with an invariant Log-Laplace transform c. The conditional Laplace transform tends to a limit independent of the conditioning variable if and only if:*

$$\lim_{h\to\infty} a^{\circ h}(u) = 0, \forall u.$$

Proof: It is clear that this condition is a necessary condition. It is sufficient since:

$$\lim_{h\to\infty} c[a^{\circ h}(u)] = c[\lim_{h\to\infty} a^{\circ h}(u)] = c(0) = 0.$$

QED

We note that the limit corresponds to the stationary distribution:

$$\lim_{h\to\infty} E\left[\exp(-u'Y_t) \mid Y_{t-h}\right] = \exp\left[c\left(u\right)\right],$$

which is necessarily unique.

Remark 2.1: A sufficient condition for ergodicity is easily derived in the unidimensional case, when the function a is the Log-Laplace transform of an infinitely divisible distribution on $\mathbb{R}^+$. Indeed we know that $a(u) \geq 0$ for $u \geq 0$, satisfies $a(0) = 0$, and is increasing and concave. We note that:

$$a(u) \leq a(0) + \frac{da}{du}(0) \cdot u = \frac{da}{du}(0) \cdot u, \text{ for } u \geq 0,$$

where $\frac{da}{du}(0) \geq 0$. Therefore if $\frac{da}{du}(0) < 1$, the solution of the recursive equation $u_h = a(u_{h-1})$ tends to zero by the Lipschitz condition for any initial condition u_0. This is equivalent to:

$$\lim_{h\to\infty} a^{\circ h}(u) = 0, \forall u \geq 0.$$

Example 2.5 *: For a gaussian autoregressive process the conditional distribution at horizon h is* $N[\rho^h Y_{t-h}, (1-2\rho^{2h})/(1-\rho^2)] = N[\rho^h Y_{t-h}, 1+\rho^2+\cdots+\rho^{2(h-1)}]$. *We see that* $a^{\circ h}(u)=\rho^h u$, $c(u)-c[a^{\circ h}(u)] = \left[\frac{1}{1-\rho^2} - \frac{\rho^{2h}}{1-\rho^2}\right]\frac{u^2}{2}$, $\sum_{k=0}^{h-1} b[a^{\circ k}(u)]= \left(\sum_{k=0}^{h-1}\rho^{2h}\right)\frac{u^2}{2}$.

2.5 Invariance by aggregation

The class of *CAR(1)* processes is invariant by aggregation. Indeed let us consider independent processes $Y_{j,t}$, $j = 1, ..., J$, whose conditional Laplace transforms are given by:

$$E\left[\exp(-u'Y_{j,t}) \mid \underline{Y_{j,t-1}}\right] = \exp\left[-a\left(u\right)' Y_{j,t-1} + b_j\left(u\right)\right],$$

and denote $Y_t = \sum_{j=1}^{J} Y_{j,t}$ the aggregated process. We have:

$$\begin{aligned}
E\left[\exp(-u'Y_t) \mid \underline{Y_{j,t-1}}, j = 1, ..., J\right] &= \prod_{j=1}^{J} E\left[\exp(-u'Y_{j,t}) \mid \underline{Y_{j,t-1}}\right] \\
&= \prod_{j=1}^{J} \exp\left[-a\left(u\right)' Y_{j,t-1} + b_j\left(u\right)\right] \\
&= \exp\left[-a\left(u\right)' \sum_{j=1}^{J} Y_{j,t-1} + \sum_{j=1}^{J} b_j\left(u\right)\right] \\
&= \exp\left[-a\left(u\right)' Y_{t-1} + \sum_{j=1}^{J} b_j\left(u\right)\right].
\end{aligned}$$

Thus the aggregated process is Markov of order one, with a *CAR* representation.

3 Analysis of the Conditional Expectation Operator

To examine more carefully the structure of temporal dependence, we derive the spectral decomposition of the conditional expactation operator [5].

For ease of exposition we consider an univariate *CAR(1)* process. We assume that this process is stationary and that Y has conditional moments at any order. We denote by $f(y_t \mid y_{t-1})$ and $f(y_t)$ the conditional and the marginal p.d.f. of the Y process, respectively.

3.1 Conditional moments

Proposition 3.1 : *We have:*

$$E\left[Y_t^n \mid Y_{t-1}\right] = P_n(Y_{t-1}), \tag{3.1}$$

where P_n is a polynomial of degree n, and its coefficient of the highest degree is:

$$\left[\frac{da}{du}(0)\right]^n . \tag{3.2}$$

Proof: We just need to identify the series expansions of:

$$E\left[\exp(-uY_t) \mid Y_{t-1}\right] = \sum_{n=0}^{\infty} \frac{u^n}{n!} E\left[Y_t^n \mid Y_{t-1}\right],$$

and of:

$$\begin{aligned}
\exp\left[-a(u)Y_{t-1} + b(u)\right] &= \sum_{n=0}^{\infty} \frac{\left[-a(u)Y_{t-1} + b(u)\right]^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{k=1}^{\infty} \left[-\frac{da^k(0)}{du^k} \frac{u^k}{k!} Y_{t-1} + \frac{db^k(0)}{du^k} \frac{u^k}{k!} \right] \right)^n,
\end{aligned}$$

to prove the proposition.

QED

In particular, the CAR processes are such that $E(Y_t|Y_{t-1}) = \alpha Y_t + \beta$, that is they satisfy a linear AR(1) model as defined by Grunwald et alii (1997).

[5] This is the discrete time analogue of the infinitesimal generator examined in continuous time models.

3.2 Spectral decomposition of the conditional expectation operator

We deduce from Proposition 3.1 the spectral properties of the conditional expectation operator.

Proposition 3.2 : *Let us consider the conditional expectation operator $\psi \to T\psi$ defined by:*

$$T\psi(y) = E\left[\psi\left(Y_t\right) \mid Y_{t-1} = y\right]. \tag{3.3}$$

This operator admits real eigenvalues $\lambda_n = \left[\frac{da}{du}(0)\right]^n$, $n \geq 0$, and the eigenfunction associated with λ_n is a polynomial of degree n, P_n say.

Proof: From Proposition 3.1, the space of polynomial functions of degree less or equal to n is invariant with respect to the conditional expectation operator. The operator restricted to this space can be represented by a diagonal matrix, with diagonal elements λ_j, $j = 0, ..., n$. The result follows directly .

QED

Thus, for a $CAR(1)$ process it is easy to predict both the exponential and power transforms of Y_t. Any transformation $\Psi(Y_t)$ can in practice be expanded using the polynomial eigenfunctions:

$$\Psi(Y_t) = \sum_{n=0}^{\infty} < \Psi, P_n > P_n(Y_t),$$

and its expectation is:

$$E[\Psi(Y_t)|\underline{Y_{t-1}}] = \sum_{n=0}^{\infty} \left[\frac{da}{du}(0)\right]^n < \Psi, P_n > P_n(Y_t).$$

Corollary 3.1 : *A necessary condition for the stationarity of an univariate CAR(1) process is:*

$$\left|\frac{da}{du}(0)\right| \leq 1.$$

Proof: Indeed the norm of the conditional expectation operator is less or equal to one.

QED

This condition has to be compared with the sufficient condition for ergodicity $\left|\frac{da}{du}(0)\right| < 1$ derived in Remark 2.1.

4 Reversible Processes

A number of financial applications in continuous time are based on one dimensional diffusion processes. It is known that these processes are reversible, that is they have the same dynamics in calendar and reversed time. To establish a link with the financial literature and point out the reversible processes with jumps, we examine below the *CAR(1)* processes which are shown to have the reversibility property.

4.1 Definition and characterization

The process Y is said to be reversible if its dynamic properties in calendar and reversed time are identical. Since the process is Markov, the reversibility condition is equivalent to the symmetry of the joint distribution of (Y_t, Y_{t-1}) with respect to both arguments. We can write this condition in terms of the joint Laplace transform. We have:

$$\begin{aligned} E\left[\exp(-uY_t - vY_{t-1})\right] &= E\left[\exp(-vY_{t-1})E\left[\exp(-uY_t) \mid Y_{t-1}\right]\right] \\ &= E\left[\exp\left(-(a(u)+v)Y_{t-1} + c(u) - c\left[a(u)\right]\right)\right] \\ &= \exp\left(c\left[a(u)+v\right] + c(u) - c\left[a(u)\right]\right) \\ &= \exp\left(\Psi(u,v)\right) \ \text{(say)}. \end{aligned}$$

Proposition 4.1 *: The CAR(1) process Y is reversible if and only if $\Psi(u,v) = c\left[a(u)+v\right] + c(u) - c\left[a(u)\right]$ is a symmetric function of u and v.*

The above condition implies some restrictions on the functions c and a (see Appendix A).

Proposition 4.2 *: When the process Y is reversible:*
i) $a(u) = \left(\frac{dc}{du}\right)^{-1}\left[\frac{da(0)}{du}\left\{\frac{dc(u)}{du} - \frac{dc(0)}{du}\right\} + \frac{dc(0)}{du}\right]$;
ii) the function $\gamma(u) = \frac{d^2c}{du^2} \circ \left(\frac{dc}{du}\right)^{-1}(u)$ *is quadratic.*

Therefore the log-Laplace transform of the marginal distribution of a reversible *CAR* process necessarily satisfies a Ricatti differential equation:

$$\frac{d^2c}{du^2}(u) = \beta_0 + \beta_1\frac{dc}{du}(u) + \beta_2\left(\frac{dc}{du}(u)\right)^2. \tag{4.1}$$

This equation will be solved in Subsection 4.3, where we will show the examples of reversible *CAR* processes.

When the marginal distribution is found, the dynamics of a reversible CAR process is characterized by a single parameter $\frac{da}{du}(0)$.

4.2 Nonlinear canonical decomposition

Proposition 4.3 : *Let us assume* $\left|\frac{da}{du}(0)\right| < 1$. *For a reversible stationary CAR(1) process, the eigenfunctions* P_n, $n \geq 0$, *of the conditional expectation operator are orthogonal with respect to the inner-product associated with the invariant distribution* f.

Proof: Let us consider the inner-product: $E\left[P_k\left(Y_t\right) P_l\left(Y_{t-1}\right)\right]$, with $k \neq l$. This quantity is equal to:

$$E\left[E\left[P_k\left(Y_t\right) \mid Y_{t-1}\right] P_l\left(Y_{t-1}\right)\right] = \lambda_k E\left[P_k\left(Y_{t-1}\right) P_l\left(Y_{t-1}\right)\right].$$

It is also equal to:

$$E\left[E\left[P_l\left(Y_{t-1}\right) \mid Y_t\right] P_k\left(Y_t\right)\right] = \lambda_l E\left[P_k\left(Y_t\right) P_l\left(Y_t\right)\right],$$

by the reversibility property. By comparing both expressions and by noting that $\lambda_k \neq \lambda_l$, if $\left|\frac{da}{du}(0)\right| < 1$, we deduce that $E\left[P_k\left(Y_t\right) P_l\left(Y_{t-1}\right)\right] = 0, \forall k \neq l$.

QED

We can now derive the nonlinear canonical decomposition of the transition probability (see [31, Lancaster (1958)]).

Proposition 4.4 : *If* $\left|\frac{da}{du}(0)\right| < 1$, *and the stationary CAR(1) process is reversible, we have:*

$$f(y_t \mid y_{t-1}) = f(y_t)\left[1 + \sum_{n=1}^{\infty}\left[\frac{da}{du}(0)\right]^n P_n(y_t) P_n(y_{t-1})\right], \qquad (4.2)$$

where P_n, *n varying, is an orthonormal basis of polynomial eigenfunctions of the conditional expectation operator.*

Proof: By Proposition 4.2 we can introduce an orthonormal basis (P_n, $n \geq 0$) of polynomial eigenfunctions of the conditional expectation operator, and write the Fourier expansion of $f(y_t \mid y_{t-1})/f(y_t)$ for any y_{t-1}, whenever this function is square integrable. We get:

$$f(y_t \mid y_{t-1}) = f(y_t)\left[\sum_{n=0}^{\infty} P_n(y_t) Q_n(y_{t-1})\right], \text{ say.}$$

Let us now compute the conditional expectation of $P_l(Y_t)$. We get:

$$E\left[P_l\left(Y_t\right) \mid Y_{t-1}\right] = \sum_{n=0}^{\infty} E\left[P_l\left(Y_t\right) P_n\left(Y_t\right)\right] Q_n(Y_{t-1}) = Q_l(Y_{t-1}),$$

since the polynomial functions P_n are orthonormal. Thus we deduce:

$$Q_l(Y_{t-1}) = \left[\frac{da}{du}(0)\right]^l P_l(Y_{t-1}).$$

QED

By recursion we derive the forecasting distributions at any horizon.

Corollary 4.1 : $f_h(y_t \mid y_{t-h}) = f(y_t)\left[1 + \sum_{n=1}^{\infty}\left[\frac{da}{du}(0)\right]^{hn} P_n(y_t)P_n(y_{t-h})\right].$

We note that the distribution of the process is characterized by the function c and the scalar $\frac{da}{du}(0)$, and that the condition $\left|\frac{da}{du}(0)\right| < 1$ is sufficient for the ergodicity of the process, i.e. to ensure that:

$$\lim_{h\to\infty} f_h(y_t \mid y_{t-h}) = f(y_t).$$

4.3 Examples

The aim of this section is to describe in detail various types of reversible processes. To do this, we solve the Ricatti equation in (4.1) [see Appendix A] to find the expression of the c function and infer the a function from Proposition 4.2.

The processes given below are distinguished according to the properties of the characteristic equation: $\beta_0 + \beta_1 x + \beta_2 x^2 = 0$, i.e. the degree 0,1,2 of the polynomial in x, and the type of roots in a polynomial of degree two. Note that the processes do not have the same domain of admissible values. Some of them take real nonnegative values, some are integer valued, and some other ones are qualitative dichotomous.

Example 4.1 :**Autoregressive gaussian process**

The gaussian processes are obtained when $\beta_1 = \beta_2 = 0$, i.e. the γ-function is constant. Then $Y_t = \rho Y_{t-1} + \varepsilon_t$, where (ε_t) is a standard gaussian white noise, and we get:

- *Conditional distribution:* $\mathcal{N}(\rho y_{t-1}, 1)$;
- *Marginal distribution:* $\mathcal{N}(0, \frac{1}{1-\rho^2})$;
- *Log-Laplace transforms:* $a(u) = u\rho$, $b(u) = \frac{u^2}{2}$, $c(u) = \frac{u^2}{2(1-\rho^2)}$;
- *Stationarity condition:* $\left|\frac{da}{du}(0)\right| = |\rho| < 1$;
- *Polynomial eigenfunctions: Hermite polynomials;*
- *Forecasting distribution at horizon h:* $\mathcal{N}(\rho^h y_{t-h}, \frac{1-\rho^{2h}}{1-\rho^2})$;
- *Compound function a:* $a^{\circ h}(u) = \rho^h u$;
- *γ-function:* $\gamma(u) = \frac{1}{1-\rho^2}$;

- *Joint log-Laplace transform:* $\Psi(u,v) = \frac{1}{2(1-\rho^2)}(u^2+v^2+2\rho uv)$.

Example 4.2 *:***Compound Poisson process**
This process is obtained when $\beta_2 = 0$, $\beta_1 \neq 0$, *i.e. when the* γ*-function is affine. Then* $Y_t = \sum_{i=1}^{Y_{t-1}} Z_{i,t} + \varepsilon_t$, *where* $Z_{i,t} \sim \mathcal{B}(1,\alpha)$, $\varepsilon_t \sim \mathcal{P}(\lambda(1-\alpha))$. *We get:*

- *Conditional distribution:* $\mathcal{B}(y_{t-1},\alpha) * \mathcal{P}(\lambda(1-\alpha))$;
- *Marginal distribution:* $\mathcal{P}(\lambda)$;
- *Log-Laplace transforms:* $a(u) = -\log[\alpha\exp(-u)+1-\alpha]$, $b(u) = -\lambda(1-\alpha)[1-\exp(-u)]$, $c(u) = -\lambda[1-\exp(-u)]$;
- *Stationarity condition:* $0 < \alpha < 1$;
- *Polynomial eigenfunctions: Charlier polynomials;*
- *Forecasting distribution at horizon h:* $\mathcal{B}(y_{t-h},\alpha^h) * \mathcal{P}(\lambda(1-\alpha^h))$;
- *Compound function a:* $a^{\circ h}(u) = -\log\left[\alpha^h\exp(-u)+1-\alpha^h\right]$;
- γ*- function:* $\gamma(u) = -u$;
- *Joint log-Laplace transform:* $\Psi(u,v) = -\lambda(2-\alpha) + \lambda\alpha\exp(-u-v)$ $+\lambda(1-\alpha)[\exp(-u)+\exp(-v)]$.

The joint distribution is known as the correlated bivariate Poisson distribution (see [5, Campbell (1934)], [6, Consoel (1952)], [37, Teicher (1954)], [13, Dwass, Teicher (1957)], [24, Griffiths et alii (1980)], [28, Johnson et alii (1997)], chap. 37), and has been used to examine the dynamics of the number of car accidents (see [15, Edwards, Gurland (1961)], [26, Hamdan, Al-Baggati (1971)]). It has been extended to moving average processes for count data [see Gourieroux, Jasiak (2001)].

Example 4.3 *:***Autoregressive gamma process** *(see [20, Gourieroux, Jasiak (2000)])*
This process is obtained when the γ*-function is quadratic and has a double root, i.e. for* $\beta_2 \neq 0$, $\beta_1^2 - 4\beta_0\beta_2 = 0$. *Then the conditional distribution* $Y_t \mid Y_{t-1}$ *is defined by:* $Y_t \mid X_t \sim \gamma(\delta + X_t)$, *and* $X_t \mid Y_{t-1} \sim \mathcal{P}(\beta Y_{t-1})$. *It is the discrete time counterpart of the Cox, Ingersoll, Ross diffusion process [Cox, Ingersoll, Ross (1985)]. We get:*

- *Conditional distribution:* $\gamma(\delta, \beta y_{t-1})$;
- *Marginal distribution:* $(1-\beta)Y_t \sim \gamma(\delta)$;

- *Log-Laplace transforms:* $a(u) = \frac{\beta u}{1+u}$, $b(u) = -\delta \log(1+u)$, $c(u) = -\delta \log(1 + \frac{u}{1-\beta})$;
- *Stationarity condition:* $\left|\frac{da}{du}(0)\right| = |\beta| < 1$;
- *Polynomial eigenfunctions: Laguerre polynomials:*

$$P_n(y) = \left[\frac{\Gamma(\delta)\Gamma(n+1)}{\Gamma(\delta+n)}\right]^{\frac{1}{2}} \sum_{k=0}^{n} \left\{(-1)^k \frac{\Gamma(\delta+n)}{\Gamma(\delta+k)\Gamma(n-k+1)} \frac{(1-\beta)^k y^k}{k!}\right\};$$

- *Forecasting distribution at horizon h:* $\frac{1-\beta}{1-\beta^h} Y_t \sim \gamma(\delta, \beta^h \frac{1-\beta}{1-\beta^h} y_{t-h})$;
- *Compound function a:* $a^{\circ h}(u) = \beta^h u \left[1 + \frac{1-\beta^h}{1-\beta} u\right]^{-1}$;
- *γ- function:* $\gamma(u) = \frac{u^2}{\delta}$;
- *Joint log-Laplace transform:* $\Psi(u,v) = -\delta \log\left[1 + \frac{uv+u+v}{1-\beta}\right]$.

Example 4.4 :Bernoulli process with switching regimes

This process is obtained when the γ-function is quadratic with two distinct real roots, i.e. for $\beta_2 \neq 0$, $\beta_1^2 - 4\beta_0\beta_2 > 0$.
The process is qualitative with two admissible values 0 *and* 1, *and corresponds to a Markov chain with two states. We get:*

- *Conditional distribution:* $\mathcal{B}(1, \alpha(1-\gamma) + \gamma y_{t-1})$;
- *Marginal distribution:* $\mathcal{B}(1, \alpha)$;
- *Log-Laplace transforms:* $a(u) = -\log\left[\frac{(1-(1-\alpha)(1-\gamma))\exp(-u) + (1-\alpha)(1-\gamma)}{\alpha(1-\gamma)\exp(-u) + 1 - \alpha(1-\gamma)}\right]$, $b(u) = \log(1 - \alpha(1-\gamma) + \alpha(1-\gamma)\exp(-u))$, $c(u) = \log(\alpha \exp(-u) + 1 - \alpha)$;
- *Stationarity condition:* $\left|\frac{da}{du}(0)\right| = |\gamma| < 1$;
- *Polynomial eigenfunctions: Two polynomials only, which are the first Krawtchouk polynomials (see[1, Abramowitz, Stegun (1970), 22.17]).*
- *Forecasting distribution at horizon h:* $\mathcal{B}(1, \alpha(1-\gamma^h) + \gamma^h y_{t-1})$;
- *Compound function a:* $a^{\circ h}(u) = -\log\left[\frac{(1-(1-\alpha)(1-\gamma^h))\exp(-u) + (1-\alpha)(1-\gamma^h)}{\alpha(1-\gamma^h)\exp(-u) + 1 - \alpha(1-\gamma^h)}\right]$;
- *γ- function:* $\gamma(u) = -u(1+u)$;
- *Joint log-Laplace transform:* $\Psi(u,v) = \log[(1-\alpha)(1-\alpha(1-\gamma)) + \alpha(1-\alpha)(1-\gamma)(\exp(-u) + \exp(-v)) + \alpha(1-(1-\alpha)(1-\gamma))\exp(-(u+v))]$.

Example 4.5 *:*
Let us finally describe a process, when the γ-function is quadratic with conjugate complex roots, i.e. for $\beta_2 \neq 0$, $\beta_1^2 - 4\beta_0\beta_2 < 0$. We get:

- *Log-Laplace transforms:* $a(u) = \arctan[\gamma \tan u]$, $b(u) = -\log\cos u + \log\cos\arctan[\gamma \tan u]$, $c(u) = -\log\cos u$*;*
- *Stationarity condition:* $\left|\frac{da}{du}(0)\right| = |\gamma| < 1$*;*
- *Compound function a:* $a^{\circ h}(u) = \arctan\left[\gamma^h \tan u\right]$*;*
- *γ- function:* $\gamma(u) = 1 + u^2$*;*
- *Joint log-Laplace transform:* $\Psi(u, v) = -\log[\cos(u+v) + (1-\gamma)\sin u \sin v]$.

5 State Space Representation

5.1 A stochastic nonlinear autogressive process

When both $\exp[-a(u)Y_{t-1}]$ and $\exp[b(u)]$ are Laplace transforms, we can introduce a nonlinear autoregressive representation of the *CAR(1)* process. Indeed let us introduce a process (Z_t) such that the conditional distribution of Z_t given Y_{t-1} admits Laplace transform $\exp[-a(u)Y_{t-1}]$ and ε_t is a variable conditionally independent of Z_t with Laplace transform $\exp[b(u)]$. Then we can write:

$$Y_t = Z_t + \varepsilon_t. \tag{5.1}$$

Moreover, we can always write Z_t as a nonlinear function of Y_{t-1} and a stochastic simulator η_t independent of ε_t. Finally we obtain:

$$Y_t = \alpha(Y_{t-1}, \eta_t) + \varepsilon_t.$$

This is a nonlinear autoregressive representation, including an additional stochastic term in the regression function.

5.2 Filtering and Smoothing

We are now interested in forecasting the processes (Z_t), (ε_t) given the observable process (Y_t). We denote by $g(z_t \mid y_{t-1})$ the conditional distribution of Z_t given Y_{t-1}, with Laplace transform $\exp[-a(u)y_{t-1}]$, and by $h(\varepsilon_t)$ the marginal distribution of the noise with Laplace transform $\exp[b(u)]$. The proposition below is proven in Appendix B.

Proposition 5.1 *: i) The variables Z_t, t varying, are independent conditional on the observable process.*

ii) The conditional distribution of Z_t given all the values of (Y_t) coincides with the conditional distribution of Z_t given Y_{t-1}, Y_t only. This filtering distribution is given by:

$$l(z_t \mid y_{t-1}, y_t) = \frac{g(z_t \mid y_{t-1}) h(y_t - z_t)}{\int g(z \mid y_{t-1}) h(y_t - z) dz}.$$

iii) The smoothing distribution of ε_t follows directly, since $\varepsilon_t = y_t - z_t$.

For example, the following distributions characterize the processes introduced in Subsection 4.3.

Example 5.1 **:Autoregressive gaussian process**

- *Conditional distribution of Z_t: point mass at ρY_{t-1};*
- *Marginal distribution of ε_t: $\mathcal{N}(0,1)$;*
- *Smoothing distribution of Z_t: point mass at ρY_{t-1};*
- *Smoothing distribution of ε_t: point mass at $Y_t - \rho Y_{t-1}$;*

Example 5.2 **:Autoregressive gamma process**

- *Conditional distribution of Z_t:*

  It is defined in two steps : $Z_t | X_t \sim \gamma(X_t)$, and $X_t | Y_{t-1} \sim \mathcal{P}[\beta Y_{t-1}]$. It is a mixture of a point mass at zero with probability $\exp(-\beta Y_{t-1})$ and a continuous distribution with p.d.f.:

$$[1 - \exp(-\beta Y_{t-1})]^{-1} \sum_{x=1}^{\infty} \left\{ \exp(-\beta Y_{t-1}) \frac{(\beta Y_{t-1})^x}{x!} \frac{1}{\Gamma(x)} \exp(-z) z^{x-1} \right\} 1_{z>0}.$$

- *Marginal distribution of ε_t: $\gamma(\delta)$, with p.d.f. : $h(\varepsilon) = \frac{1}{\Gamma(\delta)} \exp(-\varepsilon) \varepsilon^{\delta-1} 1_{\varepsilon>0}$.*
- *Smoothing distribution of Z_t: It is a mixture of a point mass at zero with probability:*

$$(Y_t^{\delta-1}/\Gamma(\delta)) \left(\sum_{x=0}^{\infty} \frac{Y_t^{\delta+x-1} (\beta Y_{t-1})^x \Gamma(x+\delta)}{x!} \right)^{-1},$$

  and a continuous distribution with the following p.d.f.:

$$\left[\sum_{x=1}^{\infty} \frac{(\beta Y_{t-1})^x Y_t^{\delta+x-1}}{x! \Gamma(x+\delta)} \Gamma(\delta) \right]^{-1} \sum_{x=1}^{\infty} \left\{ \frac{(\beta Y_{t-1})^x Y_t^{\delta+x-1}}{x! \Gamma(x)} \frac{1}{Y_t} \left(\frac{z}{Y_t} \right)^{x-1} \left(1 - \frac{z}{Y_t} \right) 1_{1>z/Y_t>0} \right\}.$$

This continuous distribution is a mixture of beta distributions, up to a homothetic function of $1/Y_t$.

Example 5.3 *:***Compound Poisson process**

- *Conditional distribution of Z_t: $\mathcal{B}(Y_{t-1};\alpha)$;*
- *Marginal distribution of ε_t: $\mathcal{P}(\lambda(1-\alpha))$;*
- *Smoothing distribution of Z_t: $l(z_t \mid y_{t-1}, y_t) \propto \frac{\alpha^z(1-\alpha)^{y_{t-1}-z}[\lambda(1-\alpha)]^{y_t-z}}{z!(y_{t-1}-z)!(y_t-z)!}$, $0 \leq z \leq \min(y_{t-1}, y_t)$.*

6 Increasing the Autoregressive Order

We have already mentioned that any *CAR* model of order p can be written as a *CAR(1)* process of higher dimension. Conversely, it is easy to construct *CAR(p)* processes from the basic *CAR(1)* process. We first describe the approach, and next we discuss the stationarity condition.

6.1 Definition

Let us consider an unidimensional *CAR(1)* process, with Laplace transform:

$$E\left[\exp(-u_1 Y_t) \mid \underline{Y_{t-1}}\right] = \exp\left[-a\left(u_1\right) Y_{t-1} + b\left(u_1\right)\right],$$

where: $b\left(u_1\right) = c(u_1) - c\left[a(u_1)\right]$. We can introduce more lags by replacing Y_{t-1} by a linear combination of lagged values, $\beta_1 Y_{t-1} + \beta_2 Y_{t-2} + ... + \beta_p Y_{t-p}$ (say). Then the conditional Laplace transform becomes:

$$E\left[\exp(-u_1 Y_t) \mid \underline{Y_{t-1}}\right] = \exp\left[-a\left(u_1\right)\left[\beta_1 Y_{t-1} + \beta_2 Y_{t-2} + ... + \beta_p Y_{t-p}\right] + b\left(u_1\right)\right].$$

In this setup the p-dimensional process $\underline{Y_t} = (Y_t, ..., Y_{t-p+1})'$ is a Markov process of order one, whose conditional Laplace transform is given by:

$$\begin{aligned}
& E\left[\exp(-u_1 Y_t - u_2 Y_{t-1} + ... - u_p Y_{t-p+1}) \mid \underline{Y_{t-1}}\right] \\
= \; & \exp(-u_2 Y_{t-1} + ... - u_p Y_{t-p+1}) E\left[\exp(-u_1 Y_t) \mid \underline{Y_{t-1}}\right] \\
= \; & \exp(-u_2 Y_{t-1} + ... - u_p Y_{t-p+1}) \\
& \times \exp\left[-a\left(u_1\right)\left[\beta_1 Y_{t-1} + \beta_2 Y_{t-2} + ... + \beta_p Y_{t-p}\right] + b\left(u_1\right)\right] \\
= \; & \exp\left[-A(u)'\underline{Y_{t-1}} + B(u)\right],
\end{aligned}$$

where:

$$\begin{aligned}
A(u) &= \left[a\left(u_1\right)\beta_1 + u_2, ..., a\left(u_1\right)\beta_{p-1} + u_p, a(u_1)\beta_p\right]', \\
B(u) &= b(u_1), \\
\underline{Y_{t-1}} &= (Y_{t-1}, ..., Y_{t-p}).
\end{aligned}$$

Thus the p-dimensional process $(\underline{Y_t})$ satisfies a *CAR(1)* model.

6.2 Stationarity condition

By Corollary 2.2, the stationarity condition is $\lim_{h\to\infty} A^{\circ h}(u) = 0$, where the function A has the expression given above. This condition implies restrictions on $a(.)$, β_1, ..., β_p. To illustrate these restrictions, let us consider for example the gaussian and nonnegative processes.

i) **The gaussian process**

In the gaussian case we have $a(u_1) = u_1$ and the transformation A is linear. It corresponds to the matrix:

$$\begin{bmatrix} \beta_1 & 1 & 0 & . & 0 \\ & . & 1 & & . \\ & . & & 1 & 0 \\ & 0 & . & . & 1 \\ \beta_p & 0 & . & . & 0 \end{bmatrix}.$$

The stationarity condition $\lim_{n\to\infty} A^{\circ h}(u) = 0$ is equivalent to the condition implying that the eigenvalues of A are less than one (in absolute value), or equivalently that the roots of the autoregressive equation lie outside the unit circle.

ii) **Nonnegative processes**

Let us consider a nonnegative process. We get the proposition below:

Proposition 6.1 : *If $\beta_1, ..., \beta_p$ are nonnegative, and if the function a is bounded, increasing and concave, then the stationarity condition is:*
$\frac{da}{du}(0)\ (\beta_1 + \cdots + \beta_p) < 1.$

Proof: See Appendix C.

QED

This proposition is valid, for instance, for the autoregressive gamma process.

7 Statistical Inference

In this section we consider two models that represent the Markov process Y. In the first model, the path dependence is left unspecified, and is characterized by the unconstrained conditional Laplace transform:

$$E \exp(-uY_t | Y_{t-1}) = L(u, y) = \exp \Psi(u, y), \text{ say.}$$

The second model is a *CAR(1)*, with the conditional Laplace transform $L^o(u, y) = \exp \Psi^o(u, y) = \exp(-a(u)y + b(u))$. Thus we impose the constraint of linearity on the function Ψ with respect to the conditioning past value of y. In the sequel, this model will be referred to as the constrained one.

In this section, we first consider nonparametric estimation of the functions a and b in a *CAR(1)* process. Next, we develop two specification tests of the *CAR(1)* model. The first one compares the unconstrained and the CAR constrained Laplace transform estimators whereas the second one is based on a set of seemingly unrelated regressions, that involve power transformations of the current and lagged values.

The observations in our sample are denoted $Y_t,\ t = 1, \dots, T$.

7.1 Nonparametric estimation of the distribution of a *CAR(1)* process

Estimation of the constrained Laplace transform can be performed by nonlinear least squares. The estimator of $a(u)$ and $b(u)$ is defined by:

$$\left[\widehat{a}_T(u), \widehat{b}_T(u)\right] = \arg\min_{a,b} \sum_{t=1}^{T} \left[\exp(-uy_t) - \exp(-ay_{t-1} + b)\right]^2, \tag{7.1}$$

$u \in I$, and the constrained estimator of the Laplace transform is:

$$\widehat{L}_T^0(u, y) = \exp\left[-\widehat{a}_T(u)y + \widehat{b}_T(u)\right].$$

Under standard regularity conditions and when the *CAR(1)* model is well-specified, the estimator $[\hat{a}_T(u), \hat{b}_T(u)]'$ is consistent asymptotically normal, such that:

$$\sqrt{T}\left(\left[\begin{array}{c} \widehat{a}_T(u) \\ \widehat{b}_T(u) \end{array}\right] - \left[\begin{array}{c} a(u) \\ b(u) \end{array}\right]\right) \to \mathcal{N}[0, \Omega(u)],$$

where:

$$\Omega(u) = J(u)^{-1} I(u) J^{-1}(u),$$

and :

$$\begin{aligned}
J(u) &= E\left\{\exp[-2a(u)Y_t + 2b(u)]\left(\begin{array}{cc} Y_t^2 & -Y_t \\ -Y_t & 1 \end{array}\right)\right\}, \\
I(u) &= \sum_{h=-\infty}^{+\infty} \Gamma_h(u), \\
\Gamma_h(u) &= Cov\left\{\left(\begin{array}{c} -Y_{t-1} \\ 1 \end{array}\right) \exp[-a(u)Y_{t-1} + b(u)]\{\exp -uY_t - \exp[-a(u)Y_{t-1} + b(u)]\}\right., \\
& \left.\left(\begin{array}{c} -Y_{t-h-1} \\ 1 \end{array}\right) \exp[-a(u)Y_{t-h-1} + b(u)]\{\exp -uY_{t-h} - \exp[-a(u)Y_{t-h} + b(u)]\}\right\}.
\end{aligned}$$

Note that the pointwise rate of convergence is of order $\sqrt{T}$, that is the standard parametric rate of convergence.

7.2 Comparison of the Constrained and Unconstrained estimator of the Laplace transform

The unconstrained Laplace transform can be estimated by:

$$\widehat{L}_T(u,y) = \frac{\sum_{t=2}^{T} \exp(-uy_t) K_{h_T}(y_{t-1}-y)}{\sum_{t=2}^{T} K_{h_T}(y_{t-1}-y)}, u \in I = (u_0, u_1), y \text{ varying}, \tag{7.2}$$

where $K_{h_T}(v) = K(v/h_T)/h_T$, h_T is the bandwidth and K is a positive kernel. $\widehat{L}_T(u,y)$ is the Nadaraya-Watson estimator of the regression function $L(u,y)$ (see [33, Nadaraya (1964)], [38, Watson (1964)]), with well-known asymptotic properties. Under standard regularity conditions and if the interval $I = (u_0, u_1)$ is strictly included in the support of the true conditional Laplace transform, this estimator is consistent, asymptotically normal, such that:

$$\sqrt{Th_T}\left[\widehat{L}_T(u,y) - L(u,y)\right] \to \mathcal{N}[0, \Sigma(u,y)], \tag{7.3}$$

where:

$$\begin{aligned} \Sigma(u,y) &= \frac{1}{f(y)} \int K^2(v)dv\, V\left[\exp(-uY_t) \mid Y_{t-1} = y\right] \\ &= \frac{L(2u,y) - [L(u,y)]^2}{f(y)} \int K^2(v)dv, \end{aligned} \tag{7.4}$$

and f is the marginal p.d.f. of Y_t.
Following a commonly used procedure, the goodness of fit can be assessed by considering the functional residual plots of either $\widehat{L}_T(u,y) - \widehat{L}^0_T(u,y)$, or $\log \widehat{L}_T(u,y) - \log \widehat{L}^0_T(u,y)$, u, y varying.

7.3 Power autoregression

Another specification test can be based on seemingly unrelated regressions (SUR) where the dependent and explanatory variables are power transformations of the observed series. More precisely, let us introduce an integer J and the seemingly unrelated regressions model, where we regress the current powers Y_t^j, $j = 1, ..., J$ on the lagged powers 1, Y_{t-1}^j, $j = 1, ..., J$. Let us denote by θ_{ij} the regression coefficient of Y_t^i on Y_{t-1}^j in these regressions. Under the *CAR(1)* specification, we know from Proposition 3.1 that the upper diagonal coefficients $\theta_{ij} = 0$, if $j > i$. Thus in practice we can estimate the *SUR* model and test the null hypothesis $H_0 : \{\theta_{ij} = 0$, if $j > i$, $i, j \in \{1, ..., J\}\}$. We denote by ξ the associated Fisher test statistic adjusted for conditional heteroscedasticity.

This approach can be extended in order to find a data transformation for which the *CAR(1)* model is well-specified. Let us consider a set of parametric transformations $g(Y_t, \gamma)$, say, and assume that $g(Y_t, \gamma_0)$ is a *CAR(1)* process. The parameter γ_0 can be estimated in the following way. For each admissible value of the γ parameter, we estimate the *SUR* model based on the powers $g(Y_t, \gamma)^j$, $j = 1, ..., J$ and compute the test statistic $\xi(\gamma)$. Then γ_0 is estimated by $\widehat{\gamma} = \arg\min_{\gamma} \xi(\gamma)$.

8 Application

There is a large body of literature on nonlinear dynamics of financial returns r_t, say. The dynamic models used in empirical research have to accommodate various observed stylized facts, including the volatility persistence, leverage effect, and the pattern of nonlinear autocorrelograms. A CAR process cannot represent the return process itself. Indeed it is known that r_t and r_{t-1} are weakly correlated, whereas a significant correlation may exist between r_t and r_{t-1}^2, due to the existence of a risk premium [This the so-called ARCH-M effect [16, Engle, Lilien, Robbins (1987)]. However it has also been observed (see [10, Ding, Granger, Engle (1993)], [9, Ding, Granger (1996)], [19, Gourieroux, Jasiak (1999)]) that the maximum of autocorrelations $|r_{t-h}|^k$ viewed as a function of the exponent k for any fixed lag h is located around $k = 1$. Therefore for any fixed lag h the autocorrelations decrease when the exponent k moves away from 1. This particular pattern of autocorrelations is typical for a reversible *CAR(1)* model of absolute values of returns $|r_t|$, due to the form of its nonlinear canonical decomposition. In this section we fit a *CAR(1)* model to a well chosen power transform of the absolute return.

8.1 Data description

We consider daily returns on the S&P 500 index over the period January 1st 1988 to December 28th 2000, obtained from the DataStream.

[Insert Figure 8.1: Returns]

This sample contains 3480 daily observations. Figure 8.1 displays the time series of returns, while Figure 8.2 shows the marginal density of the short rate, estimated by kernel smoothing. We observe a commonly encountered peak at zero, as well as heavy tails. This last feature suggests that the marginal distribution of the data is not normal.

[Insert Figure 8.2: Marginal Density of Returns]

The autocorrelogram of returns is given in Figure 8.3. We observe that the autocorrelations are not significant and conclude on the absence of linear time dependence in the series of returns.

[Insert Figure 8.3: Autocorrelogram of Returns]

Significant temporal dependence can be found in the autocorrelations of squared returns (see Figure 8.4). It reflects the so-called volatility persistence, for squared returns viewed as a proxy of volatility.

[Insert Figure 8.4: Autocorrelogram of Squared Returns]

8.2 Power autoregression

We now search for a power transformation $Y_t = |r_t|^\gamma$ of the series of absolute values of returns, such that the coefficients of seemingly unrelated power autoregressions up to $J = 3$ form a lower triangular matrix. For the interval of γ values lying between 0.2 and 1.5, we compute the test statistic $\xi(\gamma)$ of the null hypothesis $H_0 : \{\theta_{ij} = 0 \text{ if } j > i,\, i, j = 1, 2, 3\}$. The functional form of ξ is plotted in Figure 8.5, where the minimum is reached for $\hat{\gamma} = 0.8$. Note that this value is close to the autocorrelation maximizing exponent equal to one, reported in the literature.

[Insert Figure 8.5: $\zeta(\gamma)$ Statistic]

For the fixed value of $\gamma = 0.8$, the constrained regression coefficients under the null hypothesis H_0 are:

$$\begin{array}{ccccc} |r_t|^\gamma & = & \underset{(41.427)}{0.015} & + & \underset{(5.399)}{0.093|r_{t-1}|^\gamma} \end{array}$$

$$\begin{array}{ccccccc} |r_t|^\gamma & = & \underset{(12.929)}{0.3e-03} & + & \underset{(2.190)}{0.006|r_{t-1}|^\gamma} & - & \underset{(-0.050)}{0.002|r_{t-1}|^{2\gamma}} \end{array}$$

$$\begin{array}{ccccccccc} |r_t|^\gamma & = & \underset{(5.424)}{1.1e-05} & + & \underset{(0.426)}{0.1e-03|r_{t-1}|^\gamma} & + & \underset{(0.671)}{0.007|r_{t-1}|^{2\gamma}} & - & \underset{(-0.751)}{0.084|r_{t-1}|^{3\gamma}} \end{array}$$

8.3 Eigenfunctions

The previous approach can be completed by considering the three first eigenfunctions of the conditional expectation operator, and checking if they approximately correspond to polynomials of degree 1, 2, and 3, respectively (see Figures 8.5-8.7). We observe that the patterns are not entirely compatible with this assumption, since the first canonical variate is not exactly a straight line, and some asymmetry is displayed by the second one. However these features turn out to be insignificant when the pointwise confidence bands are considered. Recall that the x-axis corresponds to the values of $|r|^{0.8}$ over the interval $(0, 0.04)$ compatible with the 99% of the probability mass observed from the marginal distribution plot in Figure 8.2 and knowing that the process takes values larger than 0.032 with a small probability.

[Insert Figure 8.6: First Canonical Variate]

[Insert Figure 8.7: Second Canonical Variate]

[Insert Figure 8.8: Third Canonical Variate]

8.4 Nonparametric estimation of the conditional Laplace transform

We still consider the series $Y_t = |r|^{0.8}$.

i) **An insight on temporal dependence**

The conditional Laplace transform $L(u, y)$ has been estimated by the kernel method (7.2), and plotted in Figure 8.9. The argument u is measured on the right axis, while the conditioning value y on the left axis.

[Insert Figure 8.9: Unconstrained estimator of Laplace Transform]

ii) **Constrained estimator**

Next, the conditional Laplace transform $L^o(u, y)$ of the *CAR(1)* model is estimated according to the procedure outlined in Subsection 7.1. As expected, the estimated a and b functions display patterns suggesting their complete monotonicity [see Figures 8.10, 8.11]. The function a [resp. b] is increasing concave [resp. decreasing convex]. The confidence bounds have been derived by a block bootstrap algorithm [see 19, Kunsch (1989)]. The accuracy of the functional estimator of b (which depends on the marginal distribution) is better than the accuracy of the estimator for a (which depends on nonlinear temporal dependence). The estimated *CAR(1)* Laplace transform is illustrated in Figure 8.12.

[Insert Figure 8.10: $a(u)$ Function]

[Insert Figure 8.11: $b(u)$ Function]

[Insert Figure 8.12: (Constrained) *CAR(1)* Laplace Transform]

iii) **Comparison of the estimators**

The relative error, which represents the difference between the constrained and unconstrained estimates of Laplace transforms, is given in Figure 8.13. Its value is less than 5%, even for large y's corresponding to extreme risks observed in the past.

[Insert Figure 8.13: Residuals]

In particular the CAR(1) model is not rejected for $|r_t|^{0.8}$.

iv) **Comparison with the Cox-Ingersoll-Ross model**

Since $|r_t|^{0.8}$ may be seen as a proxy for volatility and the Cox-Ingersoll-Ross model is often used to represent the volatility dynamics, it is natural to compare the nonparametric CAR(1) estimation with an autoregressive gamma model. More precisely we calibrate the parameters β (resp. δ) to get the estimated function a (resp. b) close to $(\beta u)/(1+u)$ [resp. $-\delta \log(1+u)$] for extreme risk aversion.

The comparison between the constrained and unconstrained functions is provided in Figures 8.14 and 8.15. They show clearly that the autoregressive gamma model [that is the discrete time equivalent of the Cox-Ingersoll-Ross model] is rejected.

[Insert Figure 8.14: Constrained and Unconstrained a Function]

[Insert Figure 8.13: Constrained and Unconstrained b Function]

9 Derivative Pricing

The CAR model relies on an affine specification of the log-Laplace transform. In a continuous time framework similar models exist and are used for instance for examining the term structure of interest rates [Dai, Singleton (1999), Duffie, Kan (1996), Duffie, Pan, Singleton (1999), Lui (1997), Singleton (2000)]. It is known that these models can also be used for derivative pricing. In this section, we show that with any historical CAR dynamics it is possible to associate a risk neutral CAR-type dynamics. The change of probability is obtained by the change of parameters.

9.1 The historical distribution

For ease of exposition we consider the case of a riskfree asset with zero riskfree rate and a risky asset with price S_t at date t. We denote the geometric return by $r_{t+1} = \Delta \log S_{t+1}$.
We assume:

$r_{t+1} = Z_{t+1}|r_{t+1}|$,

where $(Z_{t+1}) = (\ sgn\ r_{t+1})$ and $(|r_{t+1}|)$ are independent processes;

$Z_{t+1}|Z_t \sim \frac{1}{2}\delta_{+1} + \frac{1}{2}\delta_{-1}$;

$\exp(-u|r_{t+1}|\ |\underline{Z_t}, \underline{r_t}) = \exp[-a^*(u)|r_t| + b^*(u)]$.

Thus the dynamics of r_{t+1} is determined by two components. The first one corresponds to a binomial tree with equal probabilities of the up and down price changes. The second one represents the size of price changes, is independent of the direction of change, and admits a CAR(1) structure.

The conditional Laplace transform of the return process is:

$$\begin{aligned}\exp \Psi_t(u) &= E_t[\exp(-uZ_{t+1}|r_{t+1}|)] \\ &= 0.5\{\exp[-a^*(u)|r_t| + b^*(u)] + \exp[-a^*(-u)|r_t| + b^*(-u)]\}.\end{aligned}$$

9.2 Risk neutral distribution

We search for a risk neutral distribution, such that the stochastic discount factor for the period $t, t+1$ admits an exponential form in r_{t+1}:

$$M_{t,t+1} = \exp(\alpha_t r_{t+1} + \beta_t), \quad \text{say.} \tag{9.1}$$

This corresponds to the standard Esscher transform [see Esscher (1932) for the definition, Buhlman et alii (1996), Gourieroux, Monfort (2001)a,b for applications to derivative pricing]. We assume that the price at t of a derivative providing the cash-flow $g(r_{t+1})$ at date $t+1$ is:

$$C_t(g) = E_t[M_{t,t+1} g(r_{t+1})] = E_t[\exp(\alpha_t r_{t+1} + \beta_t) g(r_{t+1})].$$

When the riskfree and risky assets are actively traded on the market, this pricing formula can be written for both of them. We get the arbitrage free conditions, that determine the values of α_t and β_t:

$$\begin{cases} E_t M_{t,t+1} = 1, \\ E_t(M_{t,t+1} \exp r_{t+1}) = 1, \end{cases}$$

$$\Leftrightarrow \begin{cases} E_t[\exp(\alpha_t r_{t+1} + \beta_t)] = 1, \\ E_t[\exp((\alpha_t + 1) r_{t+1} + \beta_t)] = 1, \end{cases}$$

$$\Longleftrightarrow \begin{cases} \beta_t = -\Psi_t(-\alpha_t), \\ \Psi_t(-\alpha_t) = \Psi_t - (\alpha_t + 1). \end{cases}$$

In particular α_t is the solution of:

$$\begin{aligned} &\exp[-a^*(\alpha_t)|r_t| + b^*(\alpha_t)] + \exp[-a^*(-\alpha_t)|r_t| + b^*(-\alpha_t)] \\ = \; &\exp[-a^*(\alpha_t + 1)|r_t| + b^*(\alpha_t + 1)] + \exp[-a^*(-\alpha_t - 1)|r_t| + b^*(-\alpha_t - 1)]. \end{aligned}$$

It is easy to check that the solution of this equation is $\alpha_t = -0.5$. Thus we get a unique SDF in this class, which is compatible with the arbitrage free conditions.

9.3 Application

The pricing method above is applied to the S&P data, after replacing the functions a^*, b^* by their constrained estimators. Then we deduce the corresponding values $\hat{\alpha}_t = -0.5$ and $\hat{\beta}_t = -\hat{\psi}_t(0.5)$ of the risk correcting factors. The derivative price at horizon H : $C_t(g) = E_t\{\Pi_{h=1}^H \exp(\alpha_{t+h} + \beta_{t+h-1})g(r_{t+H})]$ can be approximated by the kernel regressogram :

$$\hat{C}_t(g) = \frac{\sum_{\tau=1}^T \Pi_{h=1}^H [\exp(-0.5 r_{\tau+1} + \hat{\beta}_{\tau+h-1}) g(r_{\tau+H}) K(\frac{r_\tau - r_t}{h})}{\sum_{\tau=1}^T K(\frac{r_\tau - r_t}{h})},$$

where K is a kernel. This price depends on the current value r_t and on the residual maturity, which is fixed to $H = 21$ days, corresponding to one month of tradable days. We show in Figure 9.1 the corresponding call prices as functions of the moneyness strike and in Figure 9.2 the associated Black-Scholes implied (conditional) volatilities. The call prices were computed conditionally on the current return r_t set equal to zero and 1.2% respectively. For comparison the figures also display the Black-Scholes price computed with the annualized historical volatility of 14.7%. We observe a (skewed) smile effect typical for a stochastic volatility model, which is compatible with the assumed dynamics of $|r_t|$, when $|r_t|$ is considered as a proxy for volatility. A positive shock on $|r_t|$ implies an increase of the implied volatility.

[insert Figure 9.1: Call Prices]

[insert Figure 9.2: Black-Scholes Implied Volatilities]

Figure 9.3 displays the implied volatilities for two opposite situations of observed extreme return : $r_t = \pm 1.2\%$.

[Insert Figure 9.3: Positive and Negative Schocks]

We observe an asymmetric reaction of the market to decreasing and increasing prices. The market price of risk is higher when the observed return is negative. The implied volatility curves in Figures 9.2 and 9.3 have been derived from the data on the underlying asset and does not take into account additional data on derivatives. Implied volatility curves derived from european call (or put) prices are regularly diffused by Bloomberg, for instance.

Figure 9.4 displays the put and call implied volatilities, directly deduced from the observed derivative prices by inverting the Black-Scholes formula with respect to the volatility. They can also be interpreted as implied volatilities computed from bid and ask call prices.

[Insert Figure 9.4: Implied Call and Put Volatility]

It is easily checked that the skewed smiles are typical for the period of interest, that the difference between the bid and ask curves increases for derivatives which are out of the money, and that the curve derived from the historical returns by the CAR methodology lies generally between the bid and ask curves deduced from option prices.

Finally note that the approach to pricing has been implemented as if the S& P 500 was an asset directly traded on the market. [6] In practice this is not the case which may also explain in part the observed bid-ask spread.

10 Concluding remarks

In this paper we proposed a new class of processes with past dependent Laplace transforms. To this class of processes belong various compound processes whose conditional Laplace transforms are affine functions of lagged values, such as the autoregressive gaussian and gamma processes, and a compound Poisson process. The advantage of introducing temporal dependence into the Laplace transform is that this approach allows to represent various forms of nonlinear persistence and to derive the stationarity and ergodicity conditions, that are not always available under the traditional approach. Estimation of the compound processes can be performed either nonparametrically or parametrically. Both methods yield consistent estimators. The goodness of fit of the model can be evaluated using the functional residual plots. We illustrate this approach using data on stock returns to which we fit a Compound Autoregressive Process. We also show how CAR processes can be used for derivative pricing.

[6] There exist traded mimicking portfolios such as the SPDR (Standard & Poor Depository Receipt). However it can be checked that the dynamics of the S&P 500 and of the SPDR differ significantly, especially when taking account for nonlinear effects.

Appendices

A The Reversibility Condition

i) The reversibility condition can be written as:

$$c[a(u)+v]-c[a(u)]-\{c[a(v)+u]-c(u)\}-c(v)+c[a(v)]=0,\forall u,$$

$$\Longleftrightarrow \sum_{j=1}^{\infty}\frac{1}{j!}\left\{\frac{d^j c}{du^j}\left[a\left(u\right)\right]-\frac{d^j c}{du^j}\left(0\right)\right\}v^j=\sum_{j=1}^{\infty}\frac{1}{j!}\left\{\frac{d^j c}{du^j}\left(u\right)-\frac{d^j c}{du^j}\left(0\right)\right\}a(v)^j,$$

By identifying the coefficients of the term v^j, we deduce that there exist constants d_{jk} such that:

$$\forall j:\frac{d^j c}{du^j}\left[a\left(u\right)\right]=\sum_{k=1}^{j}d_{jk}\frac{d^k c}{du^k}\left(u\right)+d_{j0},\forall u. \tag{A.2}$$

ii) Let us first consider the condition corresponding to $j=1$. We get:

$$\frac{dc}{du}\left[a\left(u\right)\right]-\frac{dc}{du}\left(0\right)=\frac{da}{du}(0)\left[\frac{dc}{du}\left(u\right)-\frac{dc}{du}\left(0\right)\right].$$

Thus, if $\frac{dc}{du}$ is invertible, we get an expression of function a:

$$a\left(u\right)=\left(\frac{dc}{du}\right)^{-1}\left[\frac{da}{du}(0)\left[\frac{dc}{du}\left(u\right)-\frac{dc}{du}\left(0\right)\right]+\frac{dc}{du}\left(0\right)\right]. \tag{A.3}$$

iii) The condition written for $j=2$ implies a constraint on function c. Indeed this condition can be written as:

$$\begin{aligned}\frac{d^2c}{du^2}\left[a\left(u\right)\right]-\frac{d^2c}{du^2}\left(0\right) &= \left(\frac{da}{du}(0)\right)^2\left[\frac{d^2c}{du^2}\left(u\right)-\frac{d^2c}{du^2}\left(0\right)\right]\\ &\quad+\frac{d^2a}{du^2}(0)\left[\frac{dc}{du}\left(u\right)-\frac{dc}{du}\left(\right)\right],\forall u.\end{aligned}$$

If we introduce the function: $\gamma(u)=\frac{d^2c}{du^2}\circ\left(\frac{dc}{du}\right)^{-1}(u)$, the change of variable $v=\frac{dc}{du}(u)$, and use equation $(A.2)$, the condition becomes:

$$\begin{aligned}&\gamma\left[\frac{da}{du}(0)v+\frac{dc}{du}\left(0\right)\left(1-\frac{da}{du}(0)\right)\right]-\frac{d^2c}{du^2}\left(0\right) \qquad\qquad (A.4)\\ =&\left[\gamma(v)-\frac{d^2c}{du^2}\left(0\right)\right]\left(\frac{da}{du}(0)\right)^2+\frac{d^2a}{du^2}(0)\left[v-\frac{dc}{du}\left(0\right)\right],\forall v.\end{aligned}$$

Thus there exist scalars α_j, $j=1,...,4$ such that:

$$\gamma\left(\alpha_1 v+\alpha_2\right)=\alpha_1^2\gamma(v)+\alpha_3 v+\alpha_4,\forall v. \tag{A.5}$$

We deduce that:

$$\frac{d^2\gamma}{du^2}(\alpha_1 v + \alpha_2) = \frac{d^2\gamma}{du^2}(v), \forall v,$$

which is satisfied if $\frac{d^2\gamma}{du^2}$ is a constant function, or equivalently γ is a quadratic function. Thus we conclude that the function c has to satisfy a differential equation of the type:

$$\frac{d^2c}{du^2}(u) = \beta_0 + \beta_1 \frac{dc}{du}(u) + \beta_2 \left(\frac{dc}{du}(u)\right)^2. \tag{A.6}$$

We now consider the admissible solutions of the differential equation $(A.6)$.

iv) case $\beta_1 = \beta_2 = 0$

The function c is quadratic, whereas the function a is linear by $(A.3)$. We deduce:

$$c(u) = \delta_1 u + \delta_2 u^2, a(u) = \gamma_1 u,$$

and it is easily check that the joint Log-Laplace transform:

$$\Psi(u,v) = \delta_2(u^2 + v^2) + 2\delta_2\gamma_1 uv + \delta_1(u+v),$$

is symmetric in u and v. Thus we get a gaussian process with mean $m = -\delta_1$ variance $2\delta_2$ and autocorrelation $\rho = \delta_2/\delta_1$.

v) case $\beta_1 \neq 0$, $\beta_2 = 0$

By integrating the differential equation $(A.6)$, we get for the function c a necessary form:

$$c(u) = \delta_1 u + \delta_2(1 - \exp \delta_3 u),$$

and, by Equation $(A.3)$, we deduce a necessary form for the function a:

$$a(u) = \frac{1}{\delta_3} \log\left[\alpha_0 \exp(\delta_3 u) + (1 - \alpha_0)\right].$$

Then we get:

$$\begin{aligned} \Psi(u,v) &= \delta_1(u+v) + \delta_2(2-\alpha_0) - \delta_2\alpha_0 \exp[\delta_3(u+v)] \\ &\quad -\delta_2(1-\alpha_0)(\exp\delta_3 u + \exp\delta_3 v). \end{aligned}$$

It includes as a special case the joint log-Laplace transform of the compound Bernoulli process. The other cases are deduces by change of scale (effect on δ_3), location (effect on δ_2), and by convolution (effect on δ_2).

vi) case $\beta_1^2 - 4\beta_0\beta_2 = 0$, $\beta_2 \neq 0$

By integrating the differential equation $(A.6)$, we get for the function c a necessary form:

$$c(u) = \delta_1 u + \delta_2 \log(1 + \delta_3 u),$$

and by Equation $(A.3)$, a necessary form for the function a:

$$a(u) = \frac{\alpha_1 u}{1 + (1 - \alpha_1)\delta_3 u}.$$

Then the joint Log-Laplace transform is symmetric:

$$\Psi(u,v) = \delta_1(u+v) + \delta_2 \log\left\{1 + \delta_3(u+v) + \delta_3^2(1-\alpha_1)uv\right\}.$$

Up to a change of scale and location, we get the autoregressive gamma processes.

vii) case $\beta_1^2 - 4\beta_0\beta_2 > 0$, $\beta_2 \neq 0$

We get a necessary form for the function c:

$$c(u) = \delta_1 u + \delta_2 \log\left[\alpha \exp(\delta_3 u) + 1 - \alpha\right],$$

and by equation $(A.3)$, a necessary form for the function a:

$$a(u) = \frac{1}{\delta_3} \log\left[\frac{(1-(1-\alpha)(1-\gamma))\exp(\delta_3 u) + (1-\alpha)(1-\gamma)}{\alpha(1-\gamma)\exp(\delta_3 u) + 1 - \alpha(1-\gamma)}\right].$$

Then the joint Log-Laplace transform is symmetric:

$$\begin{aligned}\Psi(u,v) &= \delta_1(u+v) + \delta_2 \log[(1-\alpha)(1-\alpha(1-\gamma)) \\ &\quad +\alpha(1-\alpha)(1-\gamma)(\exp(\delta_3 u) + \exp(\delta_3 v)) \\ &\quad +\alpha(1-(1-\alpha)(1-\gamma))\exp(\delta_3(u+v))].\end{aligned}$$

This case is associated with the Bernoulli process with switching regimes.

viii) case $\beta_1^2 - 4\beta_0\beta_2 < 0$, $\beta_2 \neq 0$

We get a necessary form for the function c:

$$c(u) = \delta_1 u + \delta_2 \log\left[\cos(\delta_3 u + \delta_4)\right] - \delta_2 \log\cos(\delta_4),$$

and by equation $(A.3)$, a necessary form for the function a:

$$a(u) = \frac{1}{\delta_3}\left[\arctan\left(\gamma \tan(\delta_3 u + \delta_4) + (1-\gamma)\tan\delta_4\right) - \delta_4\right]$$

Then the joint Log-Laplace transform is symmetric:

$$\begin{aligned}\Psi(u,v) &= \delta_1(u+v) - \delta_2 \cos\delta_4 \\ &\quad +\delta_2 \log\left[\cos(\delta_{3(}u+v) + \delta_4) + (1-\gamma)\frac{\sin\delta_3 u \sin\delta_3 v}{\cos\delta_4}\right].\end{aligned}$$

B The Filtering Distribution

Let us consider the distribution of $Z_1, ..., Z_T, Y_1, ..., Y_T$ conditional to Y_0. It is given by:

$$l(z_1, ..., z_T, y_1, ..., y_T \mid y_0) = \prod_{t=1}^{T} \left[g(z_t \mid y_{t-1}) h(y_t - z_t) \right]. \tag{B.1}$$

Thus we deduce:

$$l(z_1, ..., z_T, y_1, ..., y_T \mid y_0) = \left[\prod_{t=1}^{T} \frac{g(z_t \mid y_{t-1}) h(y_t - z_t)}{\int g(z \mid y_{t-1}) h(y_t - z) dz} \right]$$

and the property follows.

C Stationarity Condition

We have to exhibit conditions, which ensure that the solution of the p-dimensional recursive system:

$$\begin{aligned} X_{1,t} &= a(X_{1,t-1})\beta_1 + X_{2,t-1}, \\ &. \\ &. \\ X_{p-1,t} &= a(X_{1,t-1})\beta_{p-1} + X_{p,t-1}, \\ X_{p,t} &= a(X_{1,t-1})\beta_p, \end{aligned}$$

tends to $(0,...,0)$, when t tends to infinity, for any admissible initial value $(X_{1,0},...,X_{p,0})'$. The system is equivalent to:

$$\begin{aligned} X_{1,t} &= X_{p,t}\frac{\beta_1}{\beta_p} + X_{2,t-1}, \\ &. \\ &. \\ X_{p-1,t} &= X_{p,t}\frac{\beta_{p-1}}{\beta_p} + X_{p,t-1} \\ X_{p,t} &= a(X_{1,t-1})\beta_p. \end{aligned}$$

If the coefficients β_1,..., β_p are nonnegative, and the function a with values in $[0,c]$, we deduce that $X_{j,t}$, $j = 1,...,p$ takes nonnegative values and that $X_{p,t}$ is always smaller than $c\beta_p$, for any nonnegative initial values.

Moreover the sequence $(X_{p,t})$ satisfies the nonlinear recursive equation:

$$X_{p,t} = a\left[X_{p,t-1}\frac{\beta_1}{\beta_p} + ... + X_{p,t-p+1}\frac{\beta_{p-1}}{\beta_p} + X_{p,t-p}\right]\beta_p.$$

A possible limiting value l of this sequence satisfies:

$$\frac{l}{\beta_p} = a\left\{\frac{l}{\beta_p}\left[\beta_1 + ... + \beta_p\right]\right\}. \tag{C.1}$$

Since $a(0) = 0$ and a is increasing concave, we distinguish two cases.

If $\frac{da}{du}(0)(\beta_1 + ... + \beta_p) < 1$, equation $(C.1)$ admits $l = 0$ as the unique nonnegative solution. Since the sequence $(X_{p,t})$ takes values in the compact set $[0, c\beta_p]$, with a unique admissible value $l = 0$, we deduce its convergence to zero.

If $\frac{da}{du}(0)(\beta_1 + ... + \beta_p) > 1$, there are two admissible nonnegative limits 0 and l^*, say; in this case we can select appropriately the initial values to get a sequence $(X_{p,t})$ constant equal to $l^* \neq 0$.

Figure 8.1: Returns

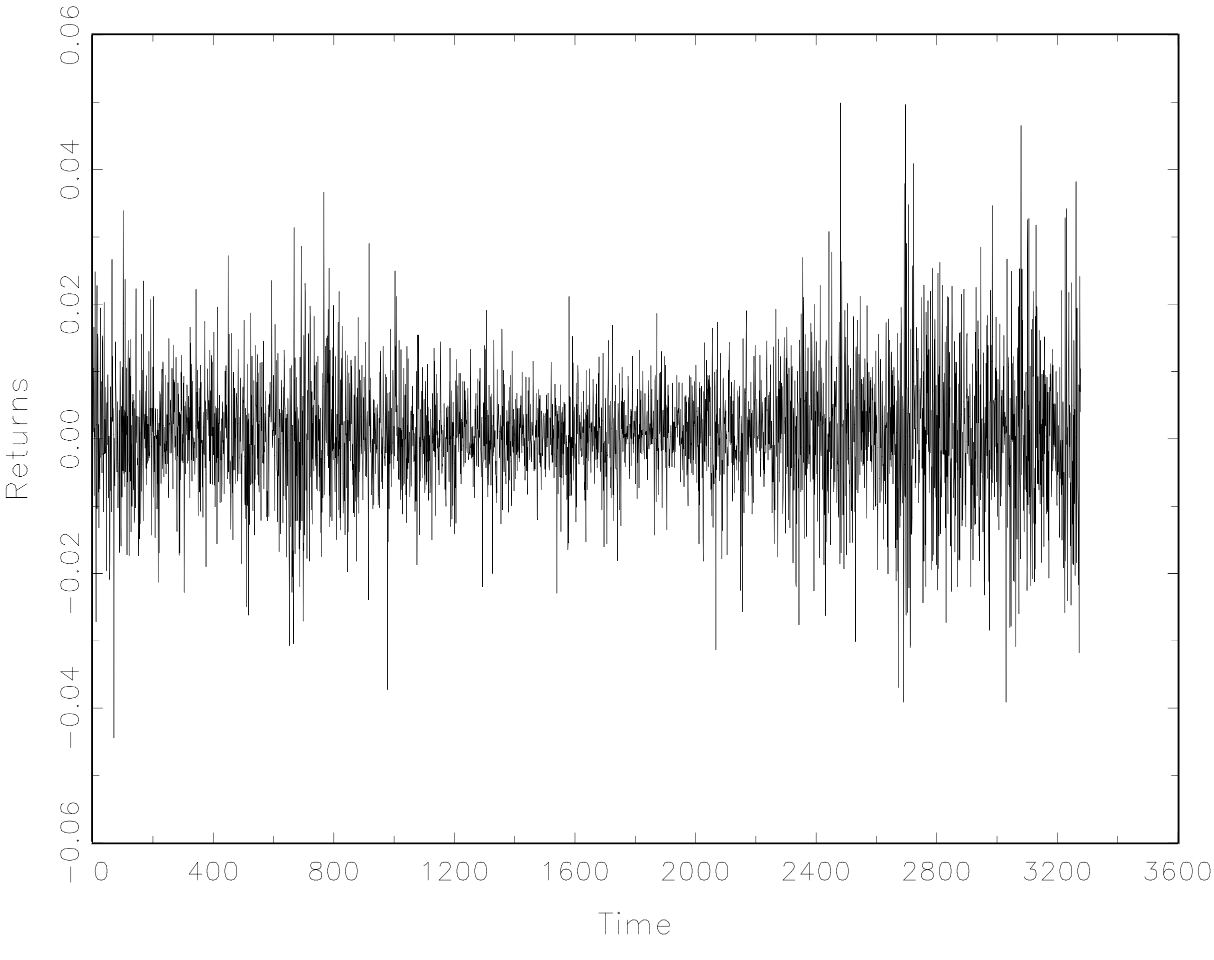


Figure 8.2: Marginal Density of Returns

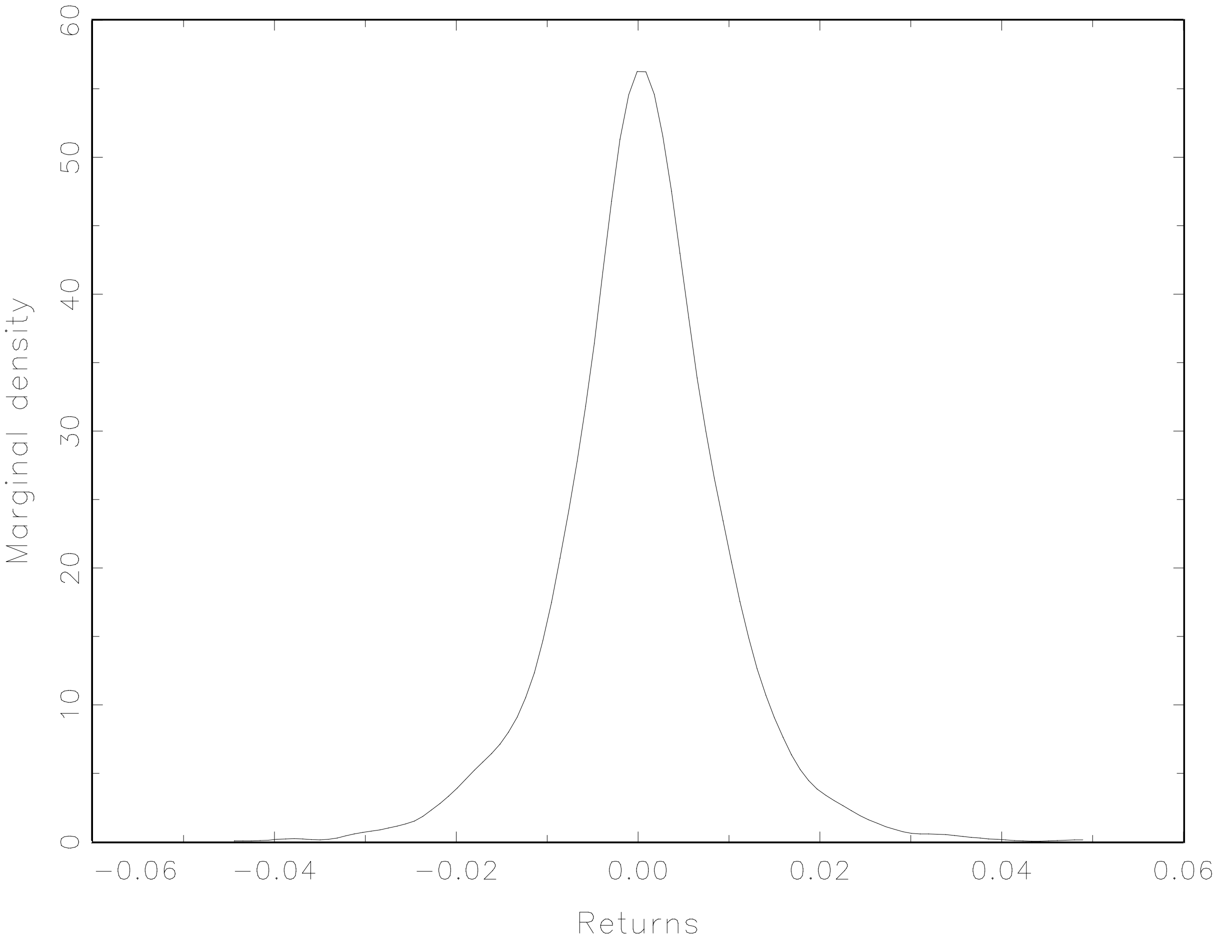


Figure 8.3: Autocorrelogram of Returns

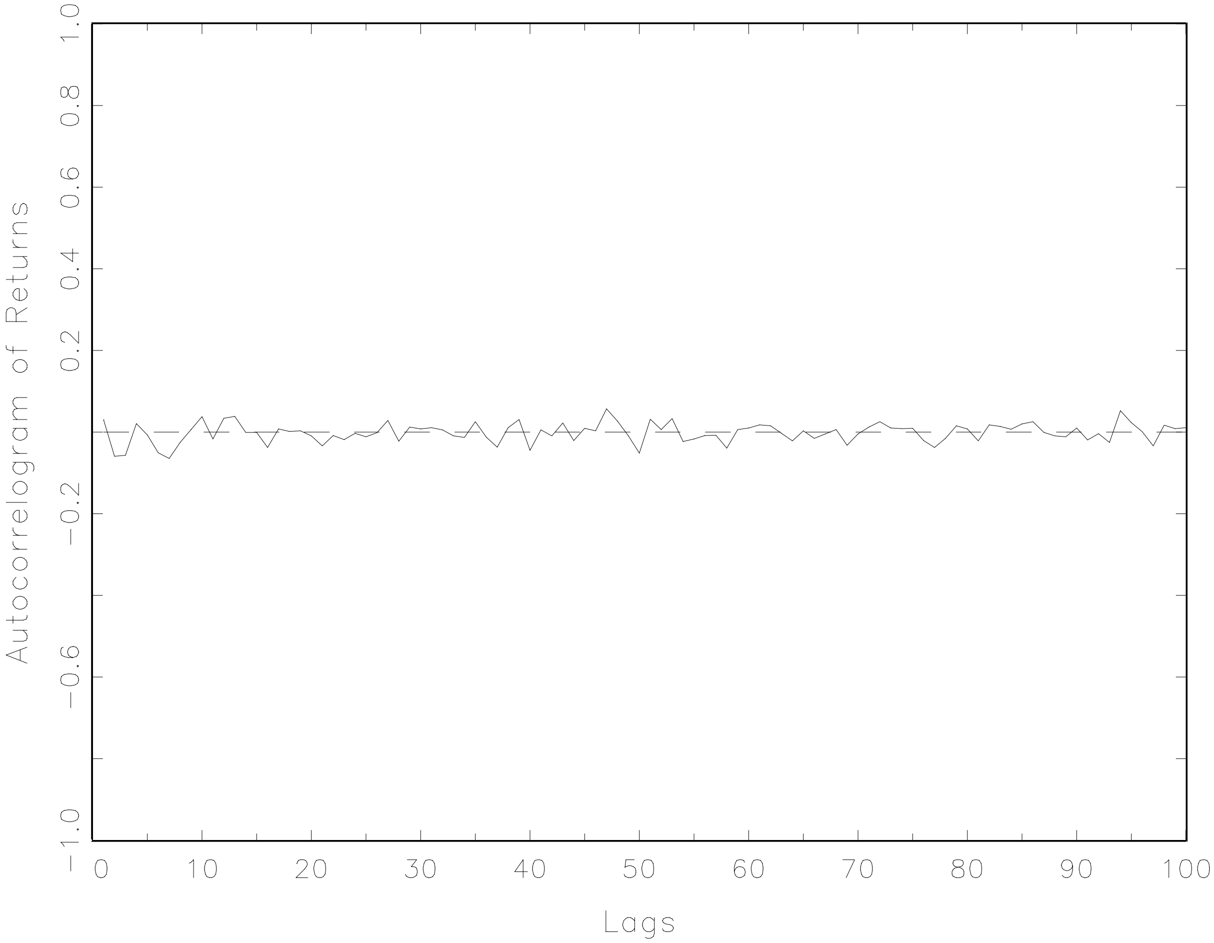


Figure 8.4: Autocorrelogram of Squared Returns

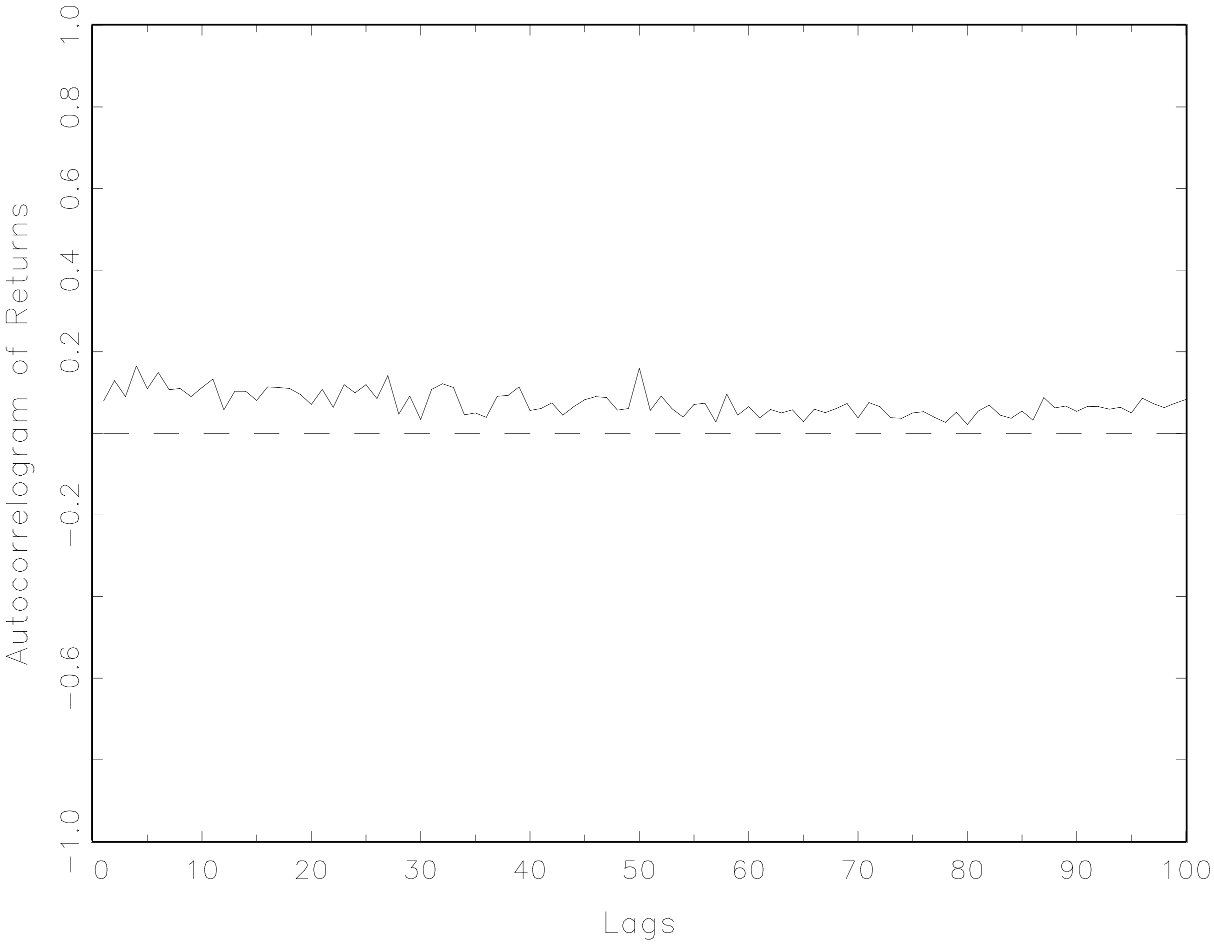


Figure 8.5: $\zeta(\gamma)$ **statistic**

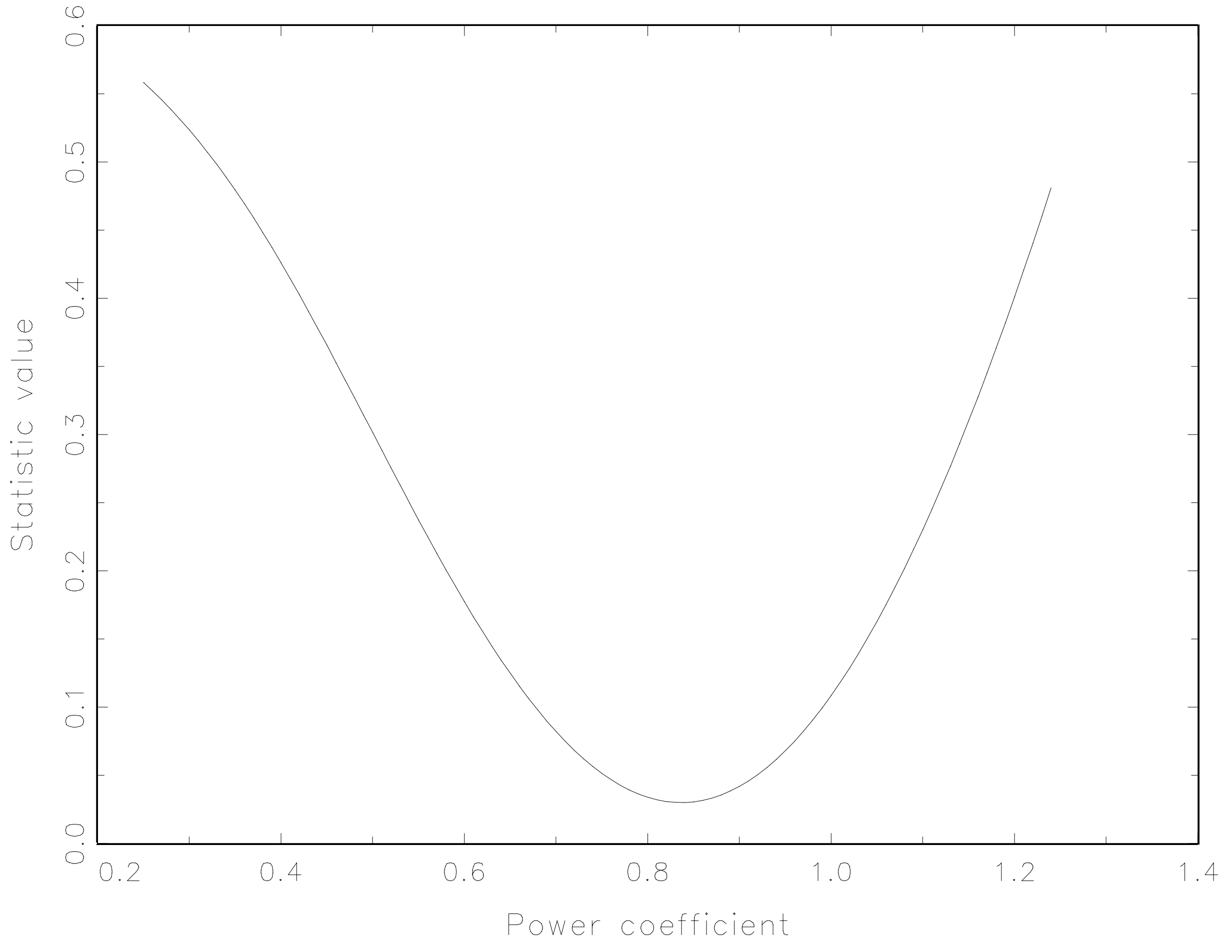


Figure 8.6: First Canonical Variate

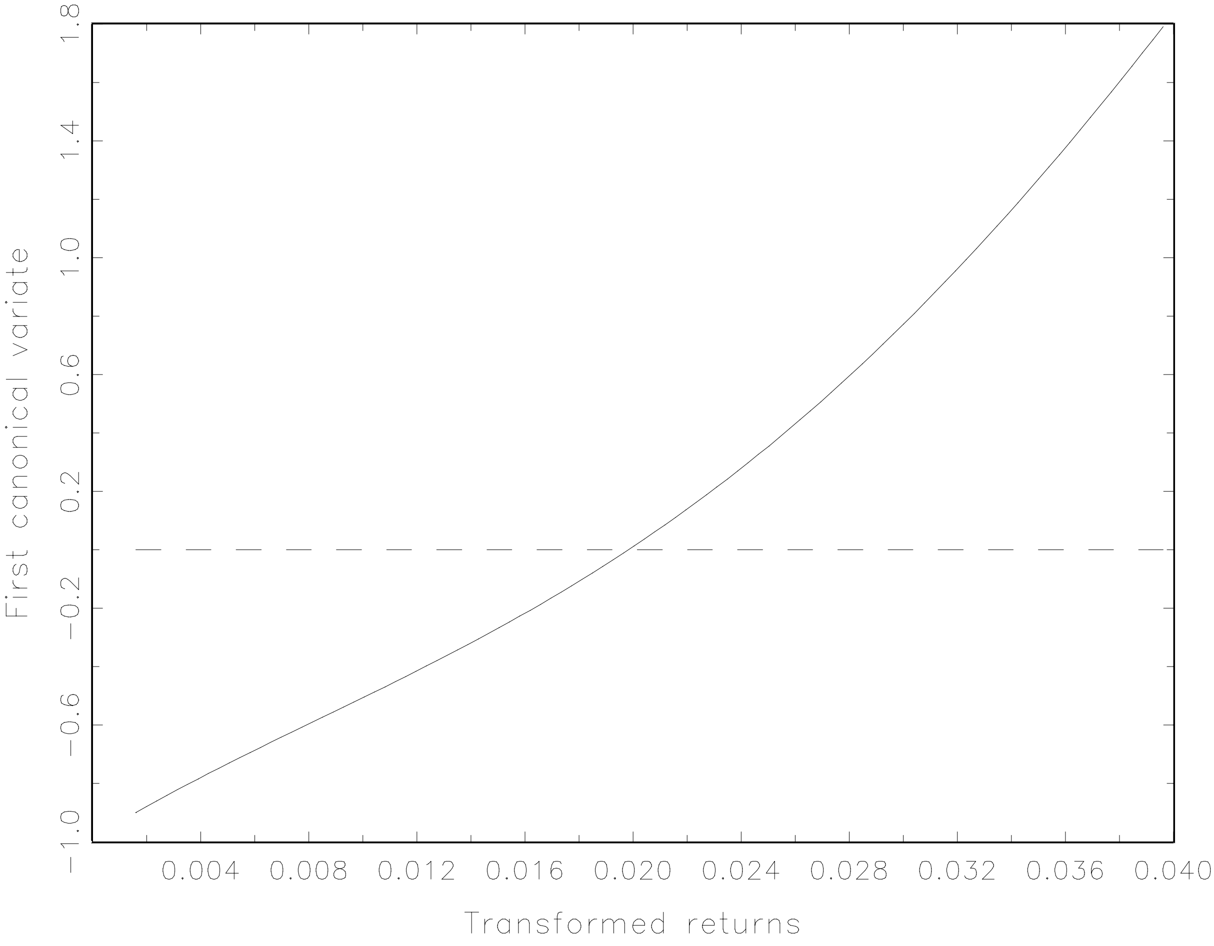


Figure 8.7: Second Canonical Variate

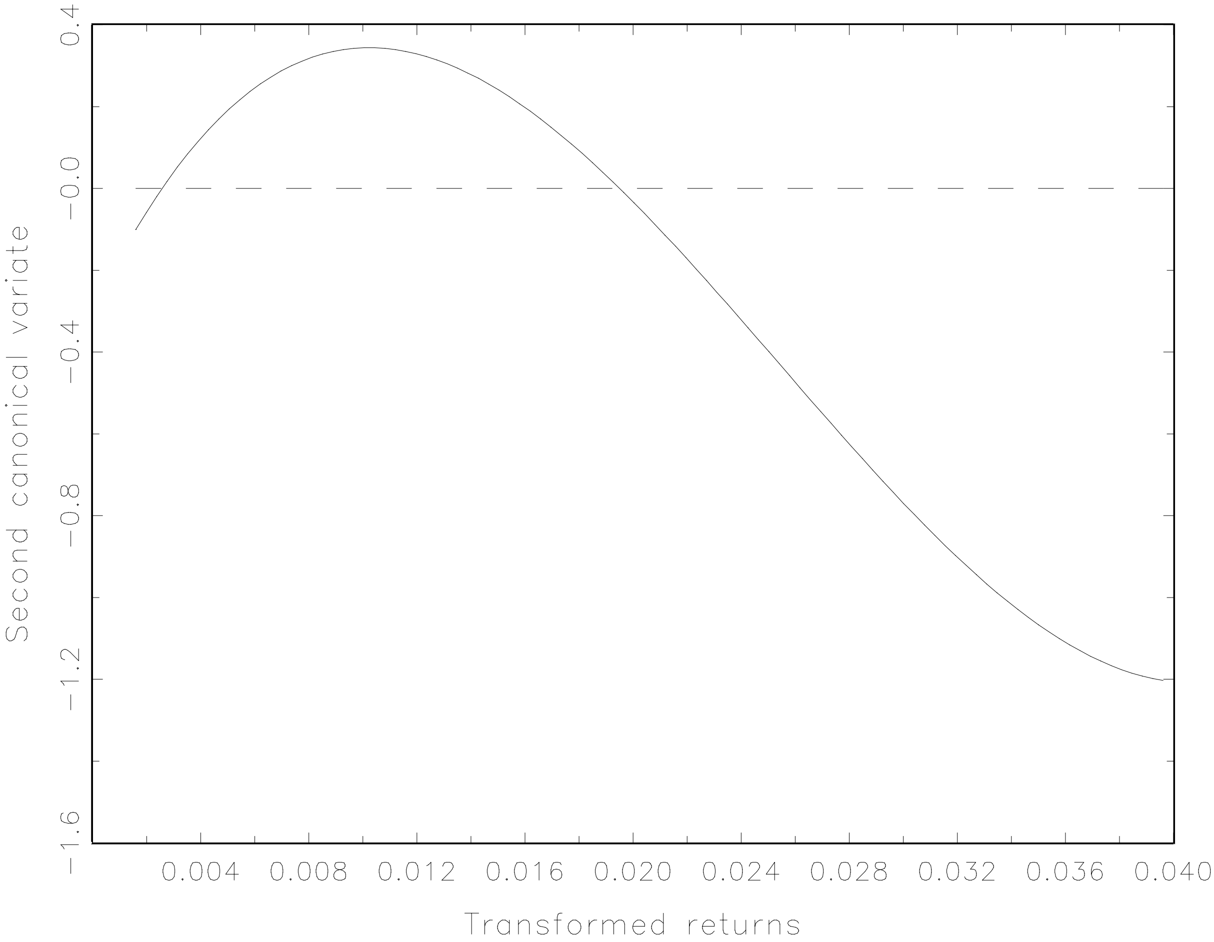


Figure 8.8: Third Canonical Variate

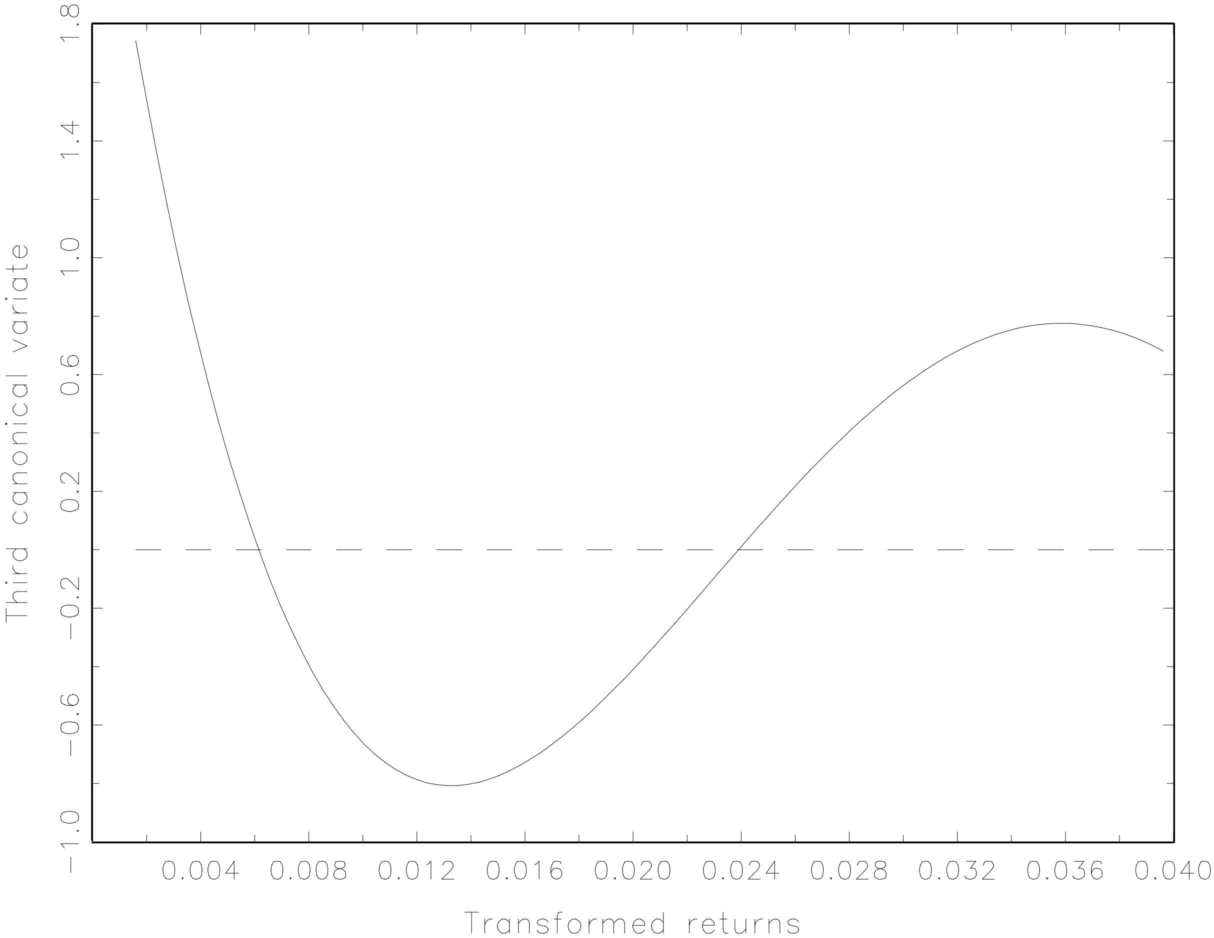


Figure 8.9: Unconstrained Estimator of Laplace Transform

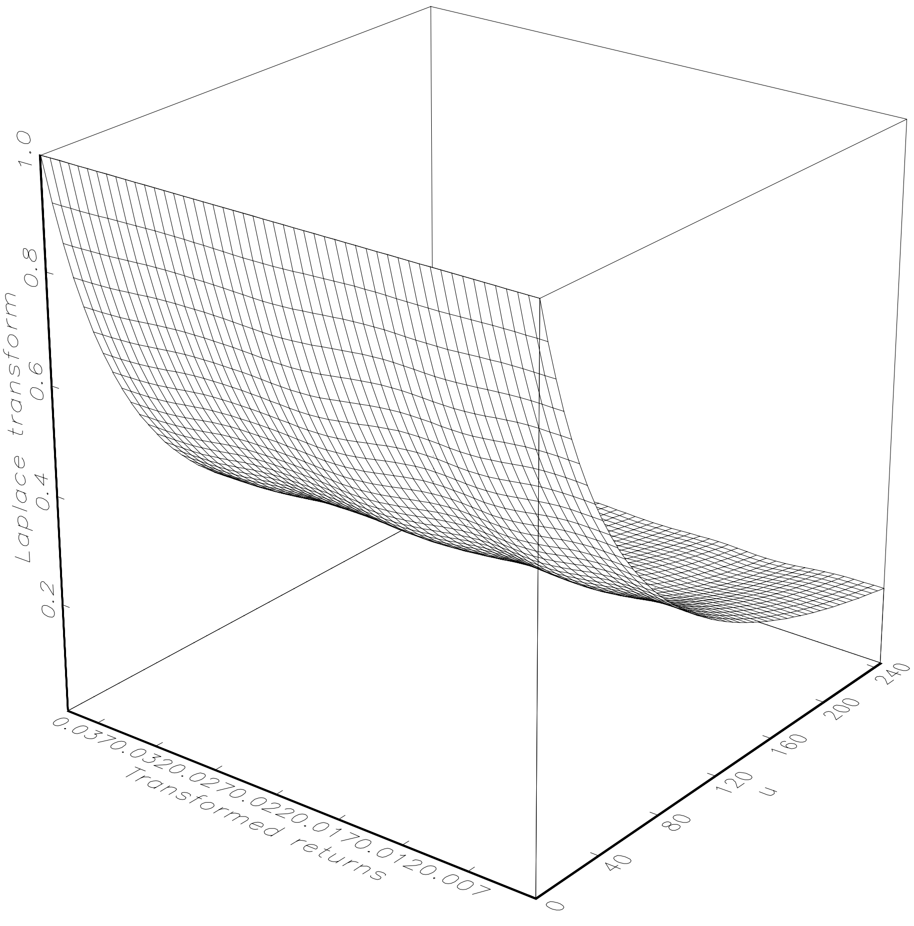


Figure 8.10: a(u) Function

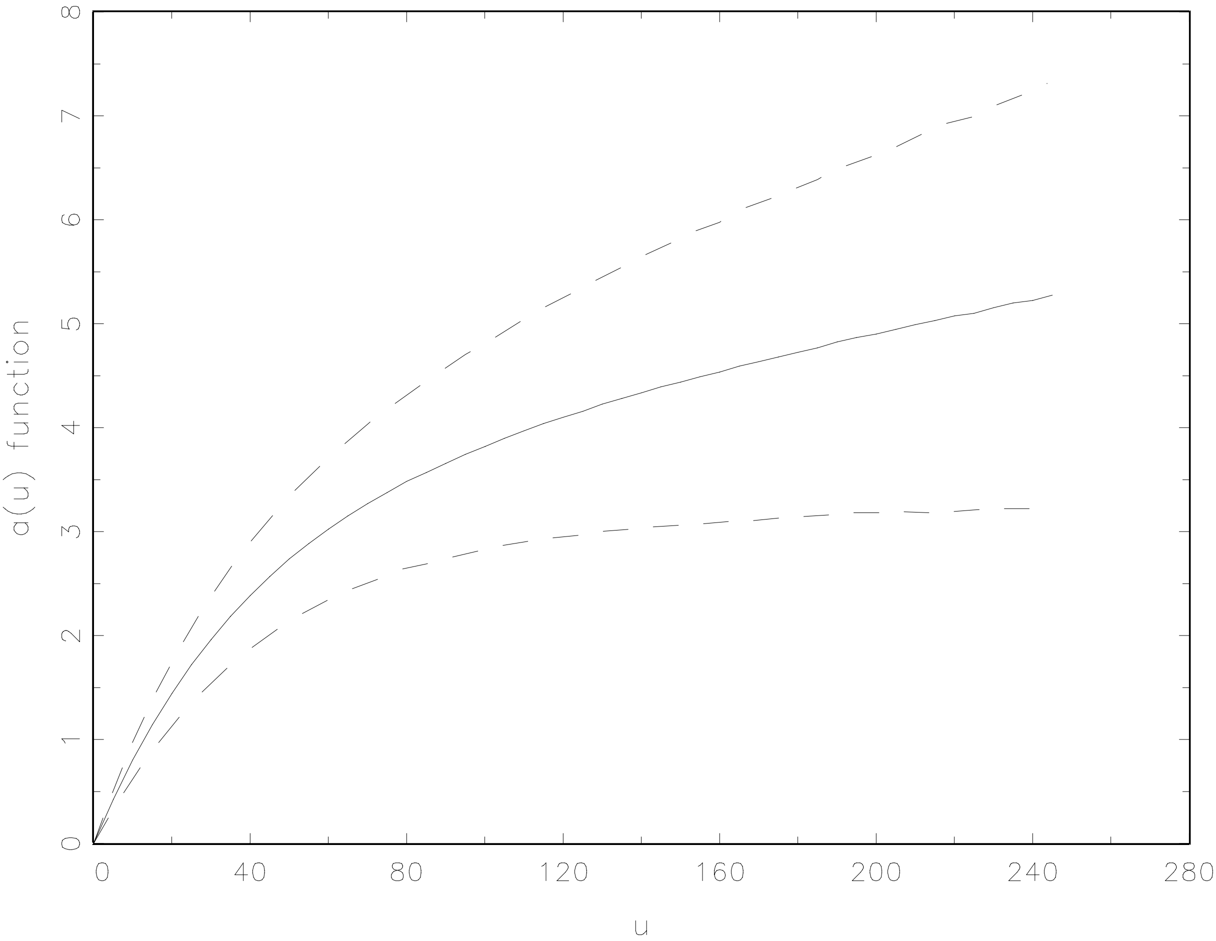


Figure 8.11: b(u) Function

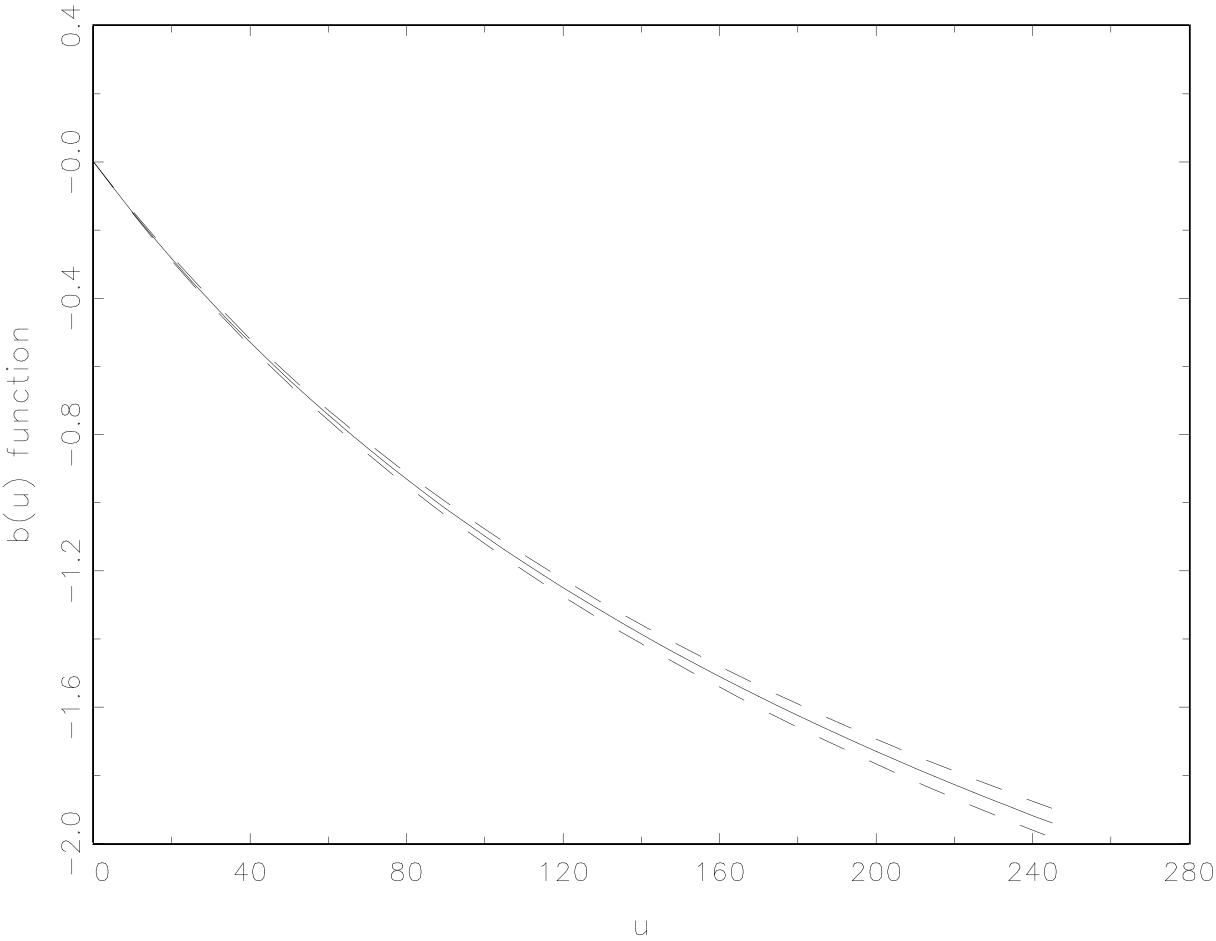


Figure 8.12: Constrained (CAR(1)) estimator of Laplace Transform

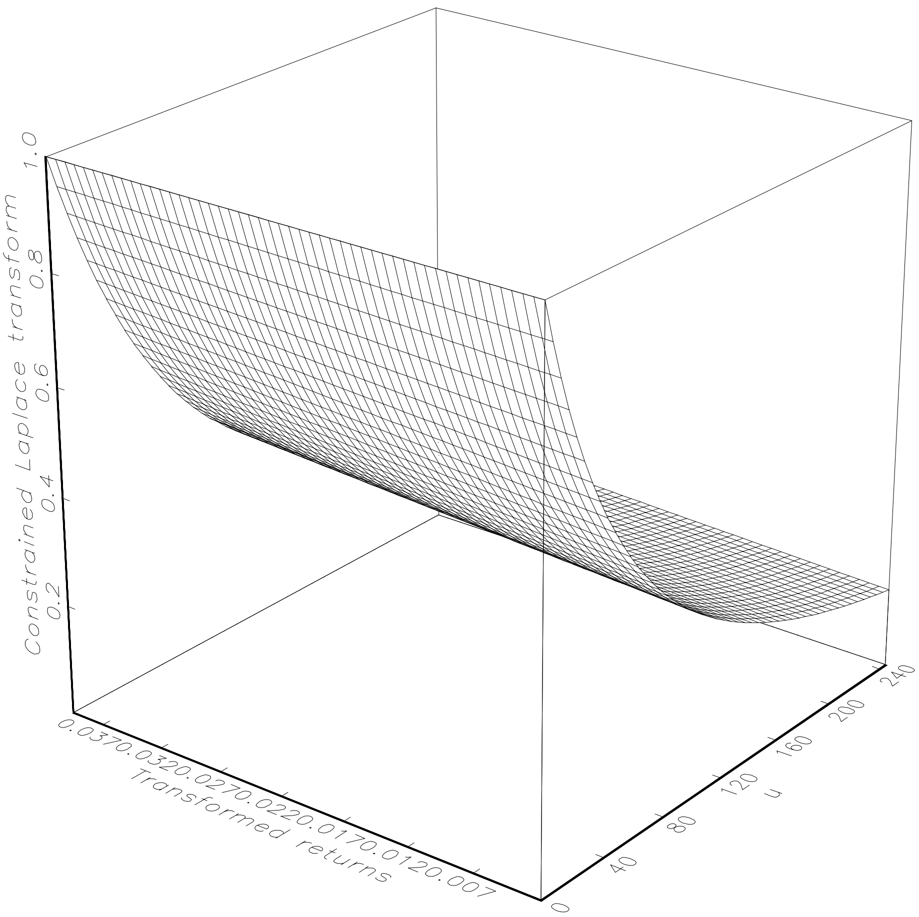


Figure 8.13: Residuals

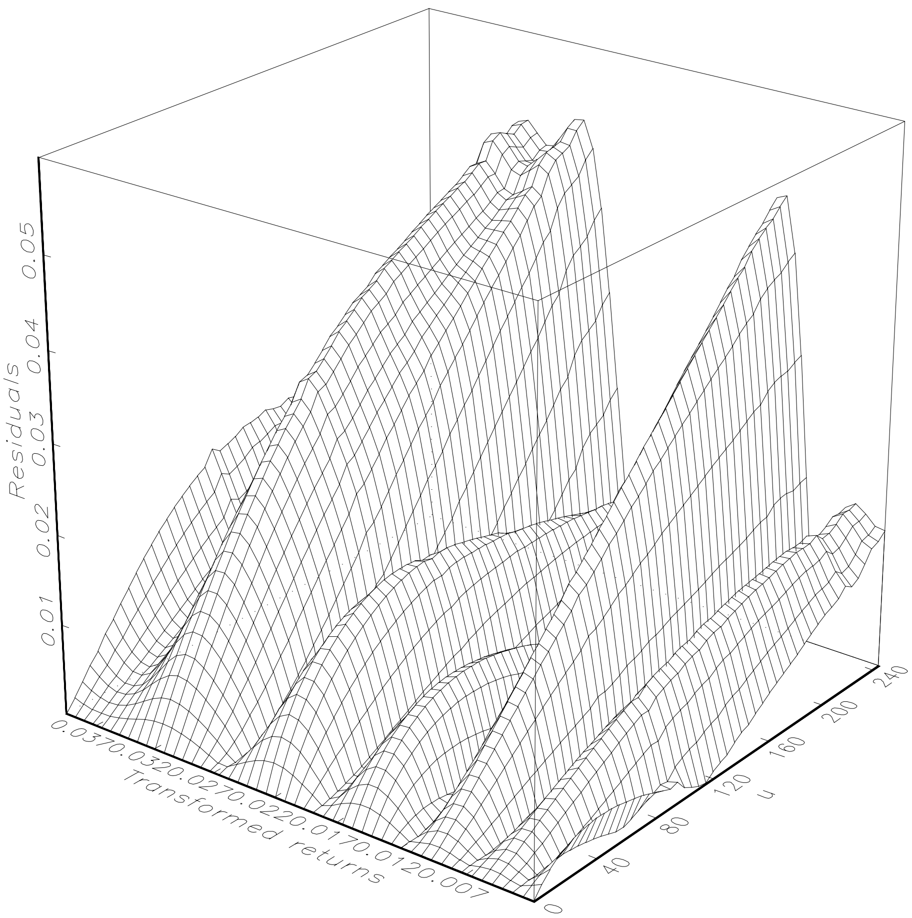


Figure 8.14: Constrained and Unconstrained a Function

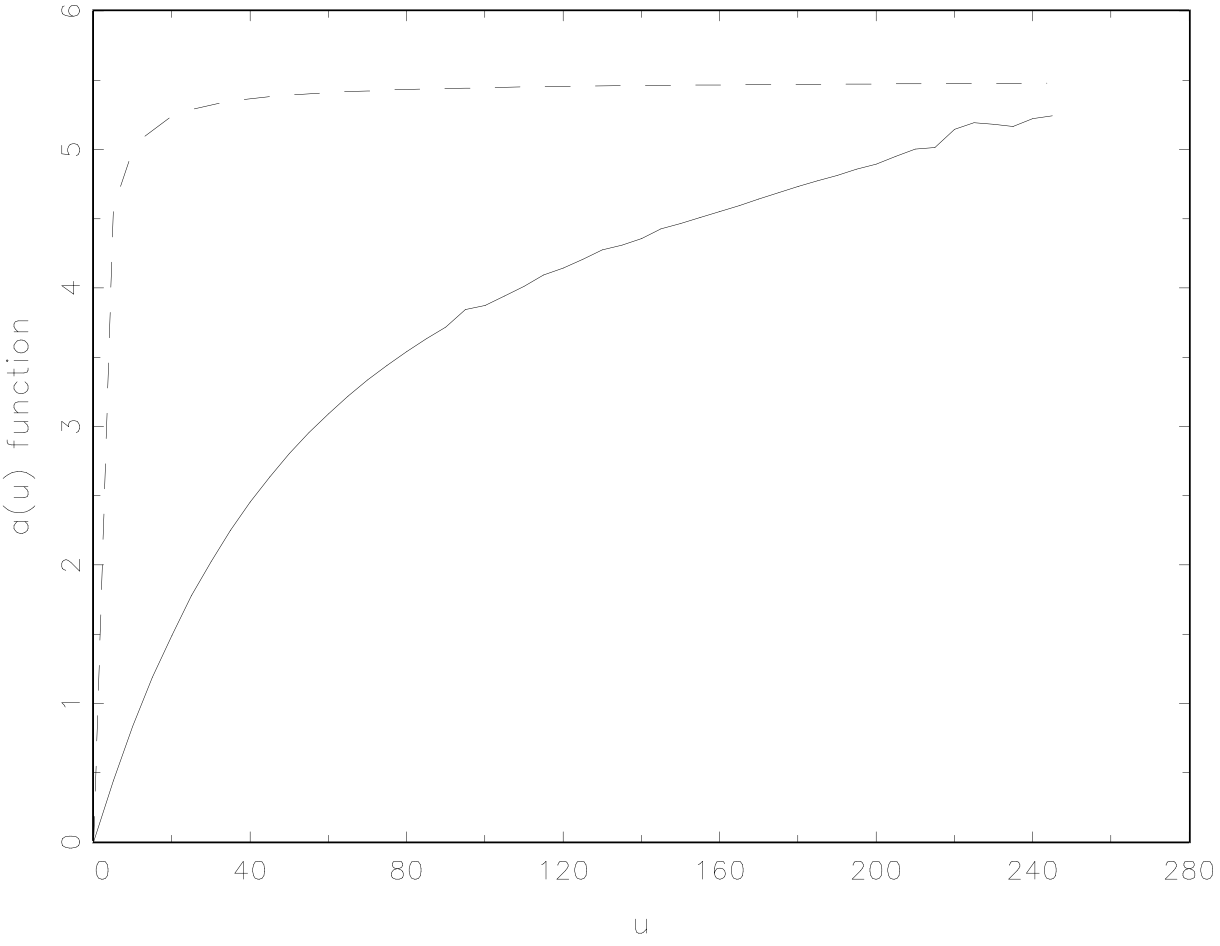


Figure 8.15: Constrained and Unconstrained b Function

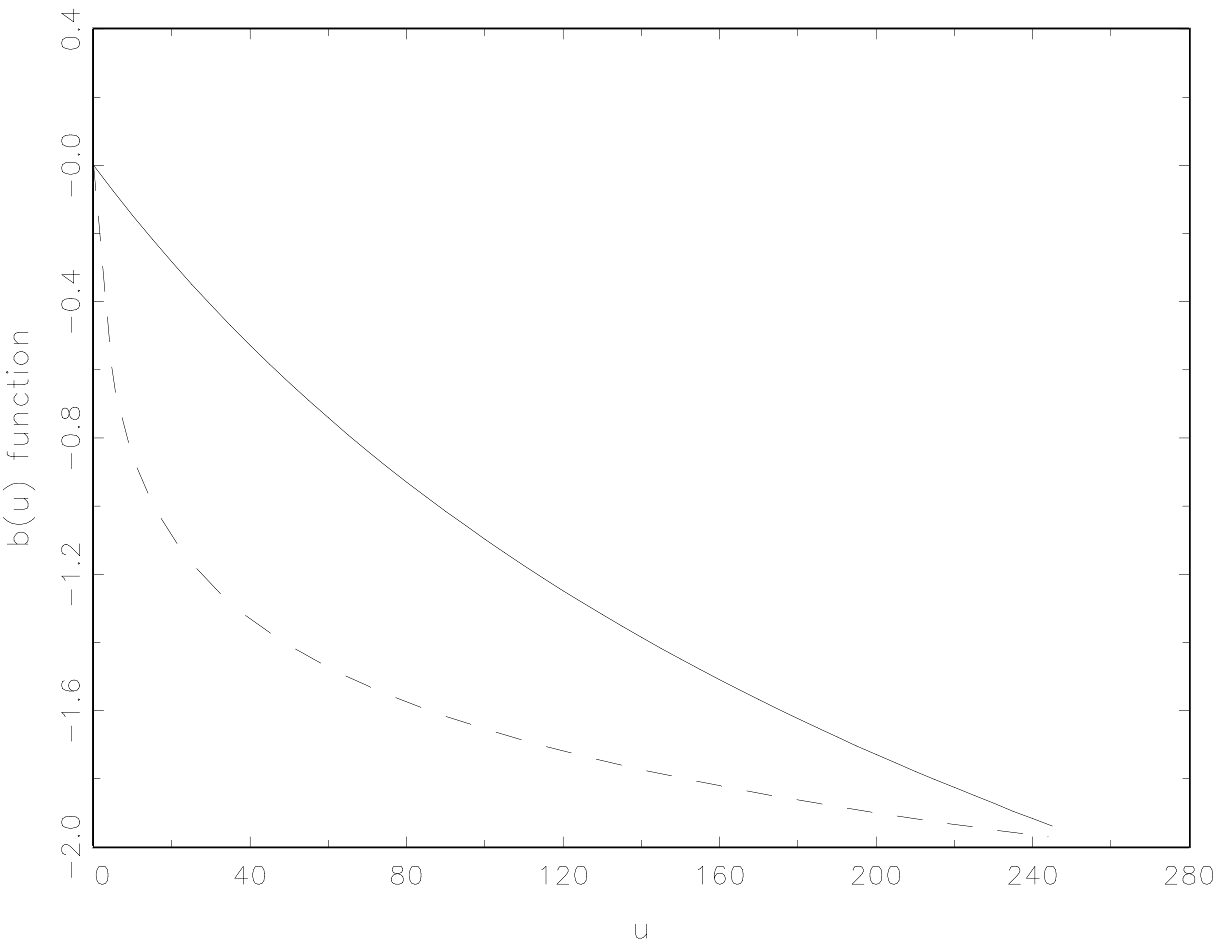


Figure 9.1: Call Prices

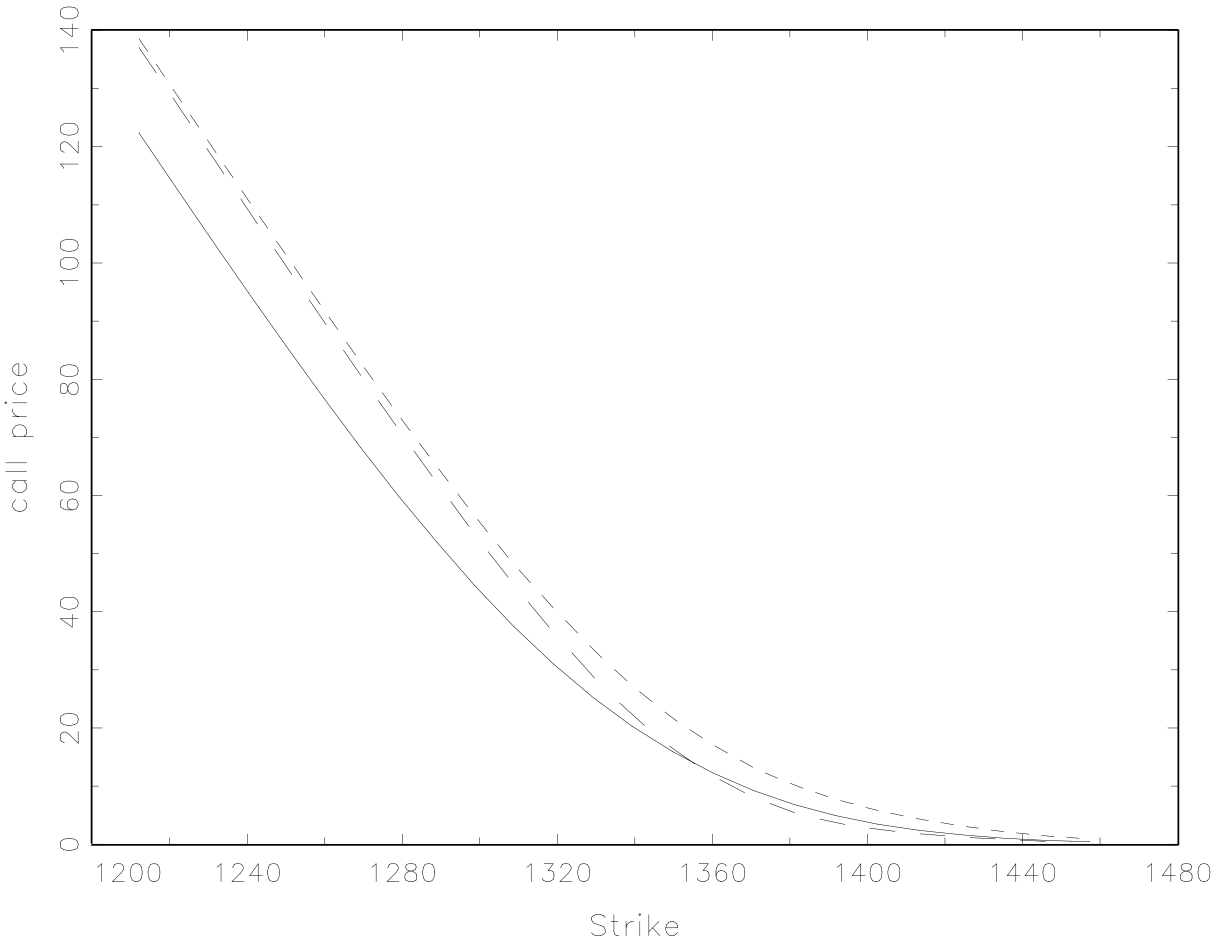


Figure 9.2: Black-Scholes Implied Volatilities

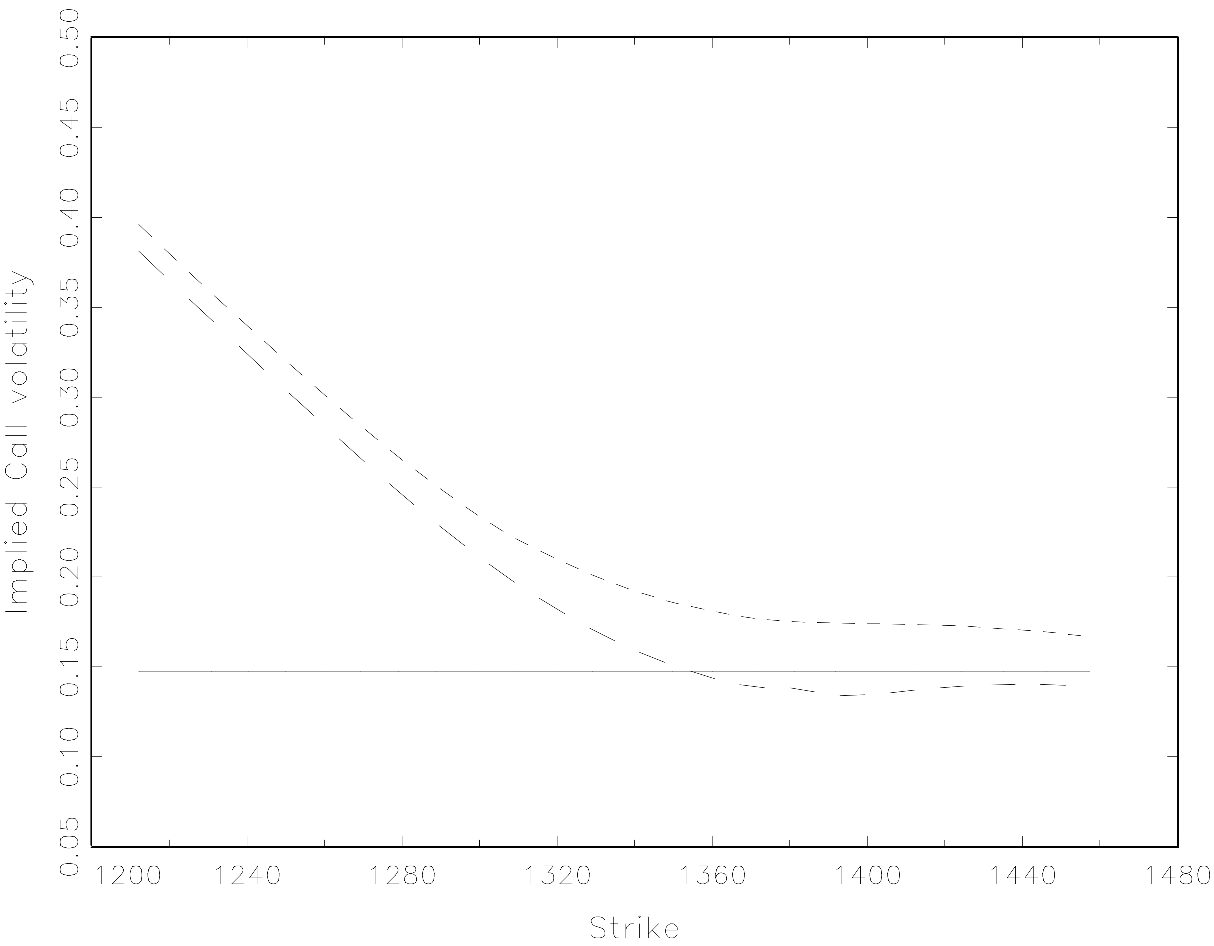


Figure 9.3: Positive and Negative Shocks

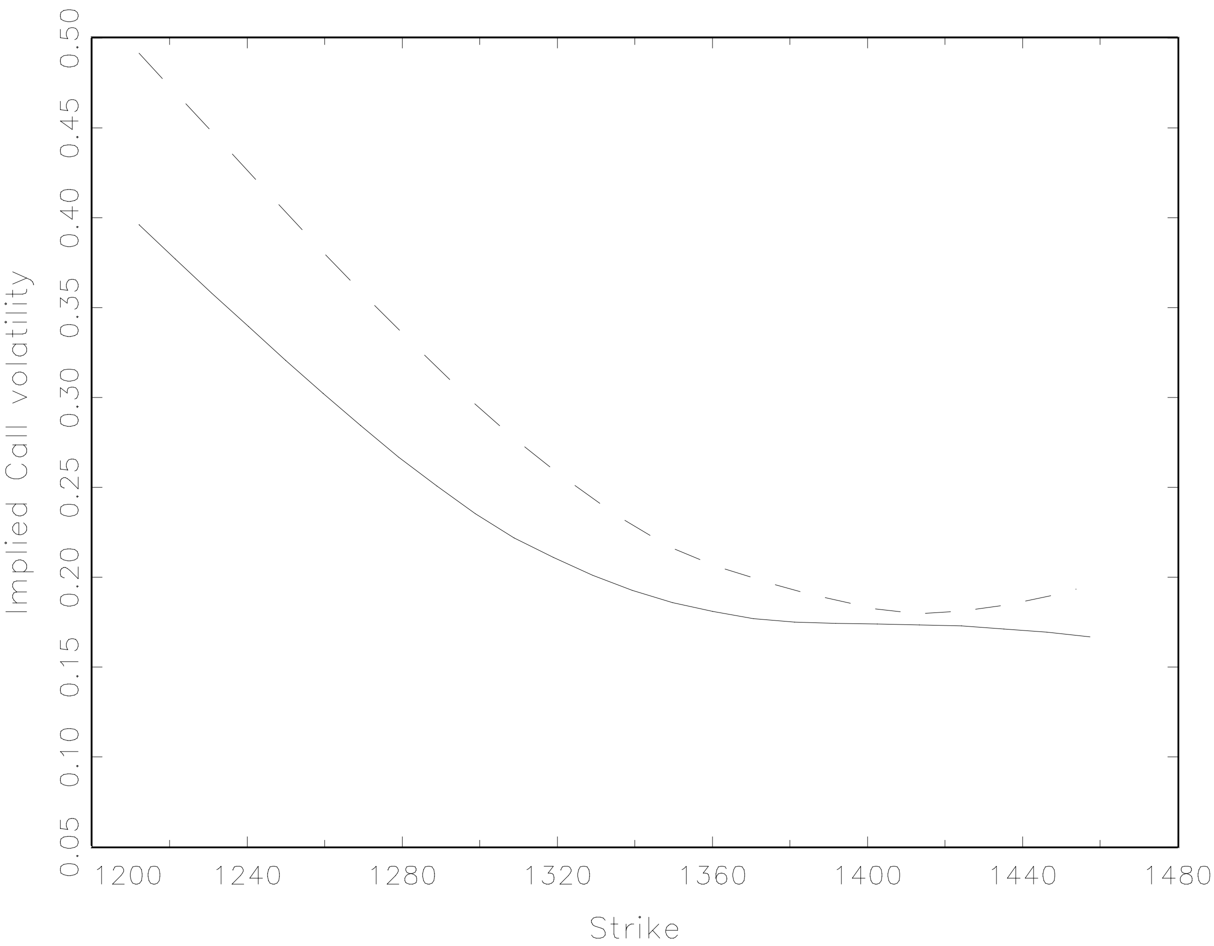
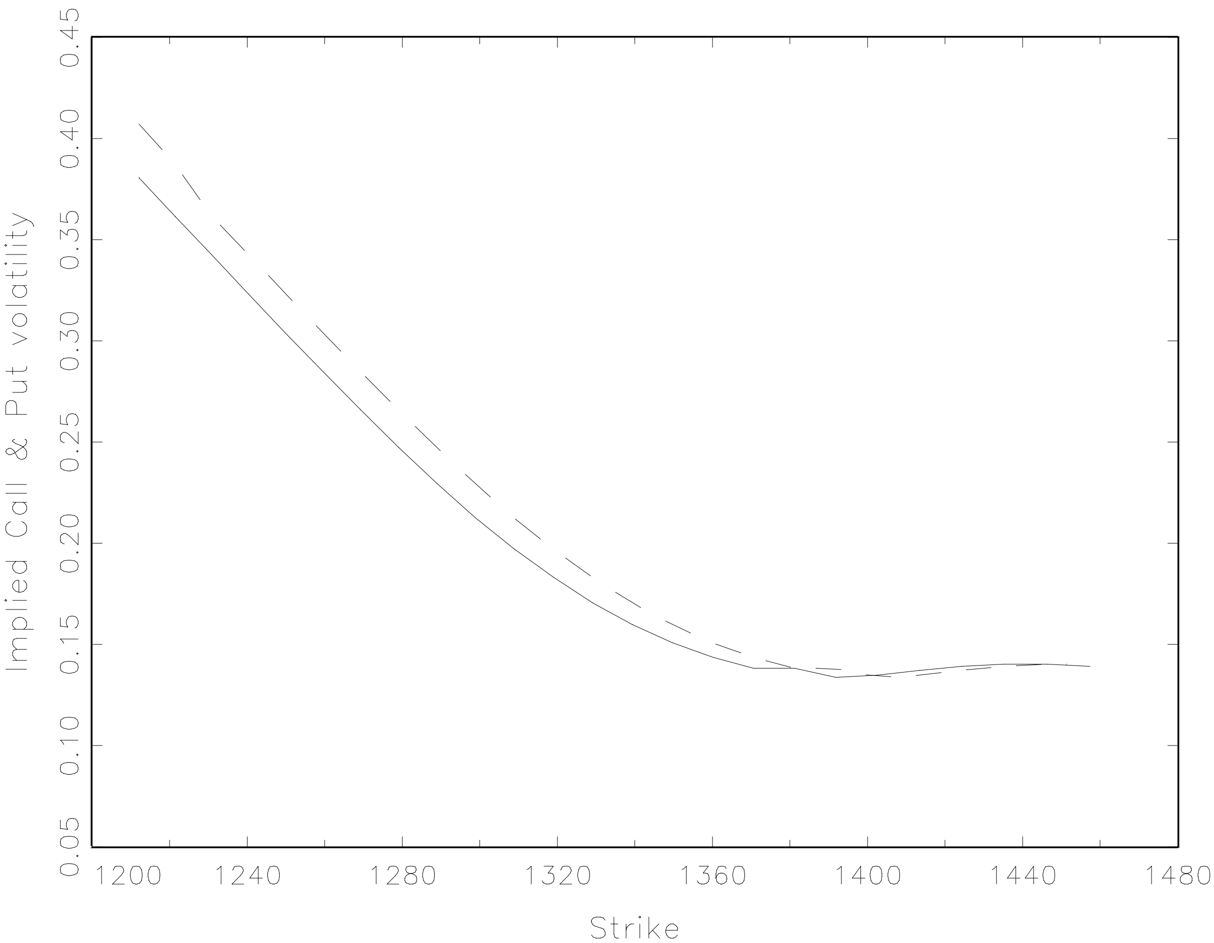


Figure 9.4: Implied Call and Put Volatility



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